TRIANGULAR NORMS IN CONNECTION WITH ENTROPY AND ENERGY OF FUZZY SETS

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A special class of t-norms is presented being suitable as norm functions for the construction of entropy and energy measures.

1. Introduction

Entropy and energy measures in the sense of DE LUCA/ TERMINI /1/, /2/ are used to compare fuzzy sets with respect to their "fuzziness". Usually, these measures are constructed via so-called norm functions. It is known from /1/ that

$F(a) = \min(a,1-a)$ for all $a \in [0,1]$

is the norm function of the special entropy measure given by the cardinality of the intersection of a fuzzy set A with its complement. Because the min - function is a special t-norm (cp. KLEMENT/6/ and MIZUMOTO/8/), the question arises whether any t-norm can be used as norm function. WEBER/9/ answers this question in the negative. Hence, we introduce in section 3 a special property named <u>concavity</u>, being sufficient for the use as norm function. Moreover, by the way, we find a new class of entropy measures being not constructable via norm functions. Before (in section 2) we sketch the concept of entropy and energy measures presented in DE LUCA/ TERMINI/1/ in a slightly generalized form. Finally, in section 4, we extend the known connection (cp. /1/) between these two measures given by the intersection of a fuzzy set with suitable other fuzzy sets to the more general case of concave t-norms instead of the min - norm function.

2. Entropy norm functions

Let X be a given universe and $\mathcal{X}$ a σ - algebra of subsets of X. Then $\tilde{\mathcal{X}}$, the set of all measurable fuzzy sets on X, is also a σ - algebra, if we adopt the ZADEH suggestions with respect to union, intersection and complement: "max", "min" and "1-". The introduction of a (σ - additive) finite measure μ on $[X, \tilde{\mathcal{X}}]$ permits us to define the (relative) cardinality of fuzzy sets.

Definition 1:
For all $A \in \tilde{\mathcal{X}}$, $\|A\|$ is called (relative) cardinality of A with respect to μ, iff $\|A\| = \mu(A)/\mu(X)$.

In the case of a finite universe $X = \{x_1, \ldots, x_n\}$, the uniform measure with $\mu(\{x_i\}) = 1/n$; $i = 1(1)n$; yields the DUBOIS/ PRADE definition (cp. /3/).
The following definitions are inspired by DE LUCA/ TERMINI /1/ and KNOPFMACHER/7/. For any $A \in \tilde{\mathcal{X}}$, let A' denote the complement of A and m_A the membership function of A.

Definition 2:
A mapping $d: \tilde{\mathcal{X}} \rightarrow [0,1]$ is called an entropy measure, iff

(i) $\forall A \in \mathcal{X}: d(A) = 0$

(ii) $\forall A \in \tilde{\mathcal{X}}: d(A) = d(A')$

(iii) $\forall A,B \in \tilde{\mathcal{X}}: A \leq' B \Longrightarrow d(A) \leq d(B)$, where

$$A \leq' B \Longleftrightarrow \forall x \in X: \begin{array}{l} m_A(x) \leq m_B(x), \text{ if } m_B(x) \leq 1/2 \\ m_A(x) \geq m_B(x), \text{ if } m_B(x) \geq 1/2. \end{array}$$

Obviously, d reaches its maximum for $U \in \tilde{\mathcal{X}}$ with $m_U \equiv 1/2$.

Definition 3:
A mapping $e: \tilde{\mathcal{X}} \rightarrow [0,1]$ is called an energy measure, iff

(i) $e(\emptyset) = 0$

(ii) $\forall A,B \in \tilde{\mathcal{X}}: A \leq B \Longrightarrow e(A) \leq e(B)$, where

$$A \leq B \Longleftrightarrow \forall x \in X: m_A(x) \leq m_B(x).$$

For the finite case these definitions essentially correspond to the original definitions of DE LUCA/ TERMINI/1/. Note, however, that in our case for the conditions (i) the converse direction is not demanded.

Hence $d(\cdot) \equiv 0$ and $e(\cdot) \equiv 0$ are also allowed as entropy and energy measures. The main advantage of the preceding definitions is that we have not to weaken the conditions for the case of an infinite universe, as in /7/.
Usually (cp. /9/), entropy measures are constructed via entropy norm functions.

Definition 4: /9/
A continous function $F: [0,1] \times [0,1] \to [0,1]$ is called an entropy norm function, iff
(i) $F(a) = 0$ for $a \in \{0,1\}$
(ii) F is increasing in $[0,1/2]$ and deccreasing in $[1/2,1]$
(iii) $\forall a \in [0,1]$: $F(a) = F(1-a)$

Note that we do not to demand the converse direction in (i).
Now we mention a result due to KNOPFMACHER/7/.

Theorem 1:
If F is a given entropy norm function then

$$d(A) = 1/\mu(X) \int F(m_A(x))\, d\mu(x) \qquad (1)$$

is an entropy measure on the measure space $[X, \mathfrak{X}, \mu]$.

Note that we use the integral in its Lebesgue - Stieltjes sense. EMPTOZ/4/ has shown that under additional assumptions all entropy measures have the representation (1).
Obviously, for example
$F_1(a) = \min(a,1-a)$ is a special entropy norm function and
$d_1(A) = \| A \cap A' \|$ is an entropy measure.
Recently, in the literature (cp. for example KLEMENT/6/, WEBER/9/, GOTTWALD/5/ and MIZUMOTO/8/) other intersection operators using so - called t-norms are discussed instead of the "min". A t-norm t is a commutative, associative, continous and monotonous mapping from $[0,1] \times [0,1]$ into $[0,1]$ with

$$t(a,1) = a \quad \text{for all } a \in [0,1]. \qquad (2)$$

For generalized union operators - so - called t-conorms s - instead of (2) it holds: $s(a,1) = 1$ for all $a \in [0,1]$.

We say that a t-norm t is <u>dual</u> to a t-conorm s, iff the generalized De - Morgan - law

$s(a,b) = 1 - t(1-a,1-b)$ for all $a,b \in [0,1]$

is valid. So we define:

$C = A \hat{\cap} B \iff \forall x \in X: m_C(x) = t(m_A(x),m_B(x))$ and

$D = A \hat{\cup} B \iff \forall x \in X: m_D(x) = s(m_A(x),m_B(x))$.

Then KLEMENT/6/ has shown that for those operators $\tilde{\mathfrak{X}}$ is also a σ - algebra. Staring from /9/ we now discuss the problem which t-norms can be used as entropy norm functions.

3. Entropy measures and t-norms

In WEBER/9/ we find a general survey on possible entropy norm functions. WEBER/9/ distinguishes 3 different types of entropy norm functions. Norm functions of type 1 are (cp. /9/) functions F with $F(a) = t(a,1-a)$ where t is a t-norm. Then t must satisfy the conditions (cp. /9/)

- $t(a,1-a) = 0$ for $a \in \{0,1\}$ (3)
- $t(a,1-a)$ is increasing for $a \in [0,1/2]$ (4)

Then, obviously, (3) is satisfied for all t-norms. But, with respect to (4), WEBER/9/ has given the following counterexample. For the t-norm

$t(a,b) = g^{-1}(g(a) + g(b))$; $a,b \in [0,1]$ with

$$g(c) = \begin{cases} 3(1/2 - c)^2 + 1/4 & \text{for } c \in [0,1/2] \\ (1 - c)^2 & \text{for } c \in [1/2,1] \end{cases}$$ it follows

$$t(a,1-a) = \begin{cases} 1/2 - \sqrt{1/4 - a + 4a^2/3} & \text{if } a \in [0,1/2] \\ 1/2 - \sqrt{7/12 - 5a/3 + 4a^2/3} & \text{if } a \in [1/2,1] \end{cases}$$

with $t(1/4,3/4) = t(1/2,1/2) = 1/2 - 1/\sqrt{12} < t(3/8,5/8) = 1/4$ in contradiction to (4).

Now we introduce:

<u>Definition 5:</u>

A t-norm t is said to be <u>concave</u>, iff the function $F(a) = t(a,1-a)$ is concave for all $a \in [0,1]$, i.e. iff

$F(\lambda a + (1-\lambda)b) \geq \lambda F(a) + (1-\lambda)F(b)$

for all $a,b,\lambda \in [0,1]$.

Lemma 1 shows that these t-norms satisfy (4).

Lemma 1: For a concave t-norm t it holds

$\forall a,b \in [0,1]$: $\min(a,1-a) \leq \min(b,1-b) \longrightarrow$
$t(a,1-a) \leq t(b,1-b)$. (5)

To prove (5), we distinguish the 4 cases $a \leq b \leq 1/2$, $1-a \leq b \leq 1/2$, $a \leq 1-b \leq 1/2$ and $1-a \leq 1-b \leq 1/2$ and use that e.g. for $a \leq b \leq 1/2$ it holds $b = \lambda a + (1-\lambda)(1-a)$ for some $\lambda \in [1/2,1]$. YAGER/10/ defines t-norms satisfying (5) as regular under complements. The concave t-norms are a special case of such norms.
Furthermore, we mention that the assumption of convexity of F, $F(\lambda a + (1-\lambda)b) \leq \lambda F(a) + (1-\lambda)F(b)$ for all $a,b,\lambda \in [0,1]$ leads to the trivial function $F \equiv 0$:
For any $\lambda \in [0,1]$ we have $a = \lambda 0 + (1-\lambda)1$ and hence $F(a) \leq \lambda F(0) + (1-\lambda)F(1) = 0$ for all $a \in [0,1]$.

We therefore obtain reasonable, i.e. non - trivial, entropy norm functions for concave t-norms.
Examples for concave t-norms are
$F_1(a) = \min(a,1-a)$ and $F_2(a) = a(1-a)$.
However, for the "bounded sum" we get
$F(a) = \max(0, a + (1-a) - 1) = 0$ for all $a \in [0,1]$.

Using concave t-norms, we define additional entropy norm functions, which are, in general, not t-norms themselves.

Lemma 2: If t and t' are concave t-norms, then
$F(a) = t(a,1-a)/(1 - t'(a,1-a))$
$F(a) = t'(a,1-a)/(1 - t(a,1-a))$
are entropy norm functions.

It is easy to check the conditions of an entropy norm function (cp. Definition 4) with the help of Lemma 1.
For $1 - t(a,1-a)$ we may also write $s(a,1-a)$ where s is the dual t-conorm to t. Then s is called a convex t-conorm, since $G(a) = s(a,1-a) = 1 - t(a,1-a) = 1 - F(a)$ is a convex function for all $a \in [0,1]$.
Hence, we obtain, for example
$F_3(a) = \min(a,1-a)/\max(a,1-a)$ and

$F_4(a) = a(1-a)/(1 - a(1-a))$ as entropy norm functions.

It is interesting that we can explain F_4 as t-norm, since $t(a,b) = ab/(a + b - ab)$ corresponds to the Hamacher - product - norm (cp. MIZUMOTO/8/). But, for F_3 this is not true. It is easy to check that $f(a,b) = \min(a,b)/\max(a,b)$ does not satisfy the monotony property of a t-norm. Moreover, this way we get back the well - known min - norm, since it holds $F(a) = ab/\max(a,b) = \min(a,b)$.

Hence, for concave t-norms t and convex t-conorms s we get

$$d_2(A) = \left\| \frac{A \,\textcircled{t}\, A'}{A \,\textcircled{s}\, A'} \right\| \tag{6}$$

as entropy measure. Note that the duality of t and s is not assumed. The entropy measure (6) have the following property.

<u>Lemma 3:</u> For all fuzzy sets A^a with constant membership grade, i.e. $m_{A^a} \equiv a$, it holds

$$d_2(A^a) = \| A^a \,\textcircled{t}\, (A^a)' \| \,/\, \| A^a \,\textcircled{s}\, (A^a)' \| \,.$$

This can easily be concluded from
$t(a,1-a)/s(a,1-a) = \int t(a,1-a)\, d\mu(x) / \int s(a,1-a)\, d\mu(x)$.
Now we can ask the question whether for any fuzzy set A

$$d(A) = \| A \,\textcircled{t}\, A' \| \,/\, \| A \,\textcircled{s}\, A' \| \tag{7}$$

is an entropy measure.

<u>Lemma 4:</u> If t is a <u>concave</u> t-norm and s a <u>convex</u> t-conorm then d(A) according to (7) is an entropy measure.

Note that from Lemma 1 we have for $A \leq' B$:
$A \,\textcircled{t}\, A' \subseteq B \,\textcircled{t}\, B'$ and $A \,\textcircled{s}\, A' \supseteq B \,\textcircled{s}\, B'$.
(6) and (7) are identical for fuzzy sets with constant membership grade and the duality of the norms is also not assumed.
Obviously, however, (7) has not a representation of the form (1) given in Theorem 1. Hence, this way we have finally obtained entropy measures of a new class which cannot be constructed by help of an entropy norm function.

4. The connection between entropy and energy

Finally, we want to extend the known connection between entropy and energy measures (cp. DE LUCA/ TERMINI/1/) to the case that general intersection operators are considered. In generalization of a result in /1/ we get:

Theorem 2:
If e is an energy measure and t a concave t-norm then
$d(A) := e(A \cap_t A')$ is an entropy measure.
Conversely, if d is an entropy measure and t a general (continous) t-norm then we have
$e(A) := d(A \cap_t U)$ with $m_U \equiv 1/2$ as energy measure.

Note that for t = min this corresponds to the result of DE LUCA/ TERMINI/1/. Using Lemma 1, we get:
$A \subseteq' B \implies A \cap_t A' \subseteq B \cap_t B'$.
The assumption of a concave t-norm made in the first statement is therefore necessary, because e.g. for the counterexample in section 3 this condition is not valid. We mention that the assumption of convexity leads to $A \cap_t A' = \emptyset$ and hence to the non - interesting entropy measure $d \equiv 0$. With respect to the second statement we know that
$A \subseteq B \implies A \cap_t U \subseteq B \cap_t U$ holds for all and not only for concave t-norms t, since t is monotone.

Summarized, the connection between the entropy and the energy of fuzzy sets is also valid for the general case of a concave t-norm as intersection operator.

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References

/1/ De Luca, A. and Termini, S., Entropy and energy measures of a fuzzy set, in: Gupta, Ragade and Yager: Advances in Fuzzy Sets Theory and Applications (North - Holland, Amsterdam, 1979, 321 - 338).

/2/ De Luca, A. and Termini, S., A definition of a non-probabilistic entropy in the setting of fuzzy sets theory, Information and Control, 20, 301 - 312, 1972.

/3/ Dubois, D. and Prade, H., Fuzzy Sets and Systems - Theory and Applications, Academic Press, 1980.

/4/ Emptoz, H., Nonprobabilistic entropies and indetermination measures in the setting of fuzzy sets theory, Fuzzy Sets and Systems, 5, 307 - 317, 1981.

/5/ Gottwald, S.; Czogota, E. and Pedrycz, W., Measures of fuzziness and operations with fuzzy sets, Stochastica, 6, 187 - 205, 1982.

/6/ Klement, E.P., Construction of fuzzy σ - algebras using triangular norms, Journal of Math. Analysis and Application, 85, 543 - 565, 1982.

/7/ Knopfmacher, J., On Measures of Fuzziness, Journal of Math. Analysis and Application, 49, 529 - 534, 1975.

/8/ Mizumoto, M., T - Norms and their pictorial representations, BUSEFAL 25, 67 - 78, 1986.

/9/ Weber, S., Measures of fuzzy sets and measures of fuzziness, Fuzzy Sets and Systems, 13, 247 - 271, 1984.

/10/ Yager, R.R., Measures of fuzziness based on t-norms, Stochastica, 6, 207 - 229, 1984.