EXAMPLES OF L-TOPOLOGIES

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1. INTRODUCTION.

In his classical paper [4] Zadeh first introduced the fundamental concept of fuzzy set. His idea was successively applied to topological spaces resulting in a study of fuzzy topology [1].

In this paper we give several examples of fuzzy topological spaces and examine the relative properties. In the sequel we assume that L is a complete lattice and that 0 and 1 are the bounds of L. If X is a set, then $F(X,L)=\{f/f:X\longrightarrow L\}$ is the class of the L-subsets of X, and $F(X)=F(X,[0,1])$. If $\alpha\in L$ the L-subset f_α is defined by setting $f_\alpha(x)=\alpha$ for every $x\in X$.

2. A GENERALIZATION OF THE INCLUDED AND EXCLUDED FUZZY TOPOLOGIES.

We start by giving a generalization of the <u>included and excluded fuzzy topologies</u> defined by E.E.Kerre [3].

<u>PROPOSITION</u> 1. Let $D=(D_x)_{x\in X}$ a family of $\vee$-complete sublattices of L, then the set:

(1) $\quad \mathcal{T}(D)=\{f\in F(X,L)/f(x)\in D_x \text{ for every } x\in X\}\cup\{f_0,f_1\}$

is an L-topology. The topologies so defined generalize the included and excluded fuzzy set topologies.

<u>Proof</u>. It is obvious that $\mathcal{T}(D)$ is an L-topology. Assume

that $L=[0,1]$ and $D_x=\{y\in L/y\geq g(x)\}$ where g is any element of $F(X)$. Then $\mathcal{T}(D)=\{f\in F(X)/f\geq g\}\cup\{f_0\}$. So we have obtained the included fuzzy set topology associated to g.

In order to obtain the excluded fuzzy set topologies we can assume that $D_x=\{y\in L/\ y\leq 1-g(x)\}$. Then $\mathcal{T}(D)=\{f\in F(X)/\ f\leq 1-g\}\cup\{f_0\}=\{f\in F(X)/1-f\geq g\}\cup\{f_0\}$.

Note that if $L=[0,1]$, then the class of closed L-subsets is $\{f\in F(X)/f(x)\in D'_x$ for every $x\in X\}\cup\{f_0,f_1\}$ where $D'_x=\{y\in L/1-y\in D_x\}$. Obviously if $D_x=D'_x$ for every $x\in X$, then $\mathcal{T}(X)$ is a <u>coplen</u> L-topology. This appens, for example, if D_x is an interval of center 1/2.

Several examples of L-topologies defined by (1) are obtained by assuming that, for every $x\in X$,

a) D_x is a finite chain;

b) $D_x=\{y\in L/f(x)\leq y\leq g(x)\}$ where $f,\ g\in F(X,L)$;

c) $D_x=\{y\in L/f(x)< y\leq g(x)\}$ where $f,\ g\in F(X,L)$;

d) D_x is a filter of L;

e) D_x is a classical topology over a set S and $L=\mathcal{P}(S)$.

3. GENERALIZED NATURAL TOPOLOGIES.

Now we will to generalize the <u>natural fuzzy topologies</u> defined by F.Conrad in [2].

<u>PROPOSITION</u> 2. Let $T=(T_j)_{j\in J}$ a family of $\vee$-complete lattices of $\mathcal{P}(X)$, and $G=(g_j)_{j\in J}$ a family of elements of $F(X,L)$. Then

the set

(2) $\mathcal{T}(G,T)=\{f\in F(X,L)/\{x\in X/f(x)\geq g_j(x)\}\in T_j$ for every $j\in J\}\cup\{f_0,f_1\}$

is an L-topology. The topologies so defined generalize the natural topologies.

Proof. Let $(f_i)_{i\in I}$ be a family of elements of $\mathcal{T}'(G,T)=\mathcal{T}(G,T)-\{f_0,f_1\}$. We have, for every $j\in J$, that

$$\{x\in X/\bigvee_{i\in I}f_i(x)\geq g_j(x)\}=\bigcup_{i\in I}\{x\in X/f_i(x)\geq g_j(x)\}\in T_j.$$

It follows that $\bigvee_{i\in I}f_i\in\mathcal{T}(G,T)$. Now, let f and g be elements of $\mathcal{T}'(G,T)$, then

$$\{x\in X/f(x)\wedge g(x)\geq g_j(x)\}=\{x\in X/f(x)\geq g_j(x)\}\cap\{x\in X/g(x)\geq g_j(x)\}\in T_j.$$

It follows that $f\wedge g\in\mathcal{T}(G,T)$ and this proves that $\mathcal{T}(G,T)$ is an L-topology. In order to obtain the natural fuzzy topology associated to a given topology S, assume that $G=(f_\alpha)_{\alpha\in[0,1]}$ and that $T_\alpha=S$ for every $\alpha\in[0,1]$. Then $f\in\mathcal{T}(G,T)$ if and only if $\{x\in X/f(x)\geq\alpha\}\in S$ for every $\alpha\in[0,1]$. This proves that $\mathcal{T}(G,T)$ is the set of lower semicontinuous maps, i.e. the natural fuzzy topology associated to S.

Interesting types of L-topologies are obtained from (2) by assuming that the T_j are filters. For example if every T_j is the filter of the cofinite subsets of X and $g_j=g$ for every $j\in J$, then $f\in\mathcal{T}(G,T)$ if and only if $f(x)\geq g(x)$ almost-everywere. Also we can generalize the construction of Proposition 2 into the following way.

PROPOSITION 3. If $T=(T_j)_{j\in J}$ and $F=(F_j^x)_{j\in J}^{x\in X}$ are families of filters of $\mathcal{P}(X)$ and L respectively, then the set

(3) $\mathcal{T}(T,F)=\{f\in F(X,L)/\{x\in X/f(x)\in F_j^x\}\in T_j$ for every $j\in J\}\cup\{f_0\}$

is an L-topology.

Proof. Let $j\in J$ and assume that $(f_i)_{i\in I}$ is a family of elements of $F(X,L)$ such that $\{x\in X/f_i(x)\in F_j^x\}\in T_j$. Then from $\{x\in X/\bigvee_{i\in I} f_i(x)\in F_j^x\}\supseteq\{x\in X/f_i(x)\in F_j^x\}$ it follows that $\{x\in X/\bigvee_{i\in I} f_i(x)\in F_j^x\}\in T_j$. Likewise, assume that $\{x\in X/f(x)\in F_j^x\}\in T_j$ and that $\{x\in X/g(x)\in F_j^x\}\in T_j$. Then $\{x\in X/f(x)\in F_j^x\}\cap\{x\in X/g(x)\in F_j^x\}\in T_j$ and from $\{x\in X/f(x)\wedge g(x)\in F_j^x\}$ $\supseteq\{x\in X/f(x)\in F_j^x\}\cap\{x\in X/g(x)\in F_j^x\}$ it follows that $\{x\in X/f(x)\wedge g(x)\in F_j^x\}\in T_j$. This proves that $\mathcal{T}(T,F)$ is an L-topology.

4. L-TOPOLOGIES ASSOCIATED TO A GRAPH.

Let (X,R) be a graph, i.e. a subset of $X\times X$. Then we can associate to (X,R) an L-topology in a very natural way.

PROPOSITION 4. Let (X,R) a graph, then the set

(4) $\mathcal{T}(R)=\{f\in F(X,L)/\ xRy \text{ implies } f(x)\leq f(y)\}$

is an L-topology.

Proof. Obviously $f_0\in\mathcal{T}(R)$ and $f_1\in\mathcal{T}(R)$. Moreover if $(f_i)_{i\in I}$ is a family of elements of $\mathcal{T}(R)$, i.e. if xRy implies that $f_i(x)\leq f_i(y)$ for every $i\in I$, it is easy to prove that xRy implies $\bigvee_{i\in I} f_i(x)\leq\bigvee_{i\in I} f_i(y)$. Then $\mathcal{T}(R)$ is $\vee$-complete. Likewise one proves that $f\in\mathcal{T}(R)$ and $g\in\mathcal{T}(R)$ imply $f\wedge g\in\mathcal{T}(R)$.

If R is an order-relation, then $\mathcal{T}(R)$ is the set of increasing maps from X to L. If G is a group of transformations on X and

(5) $R=\{(x,y)/\text{there exist } g\in G \text{ such that } y=g(x)\}$,

then $f\in\mathcal{T}(R)$ if and only if $f(x)\leq f(g(x))$ for every $x\in X$ and $g\in G$. Then it is also $f(g(x))\leq f(g^{-1}(g(x)))=f(x)$. This proves that $f\in\mathcal{T}(R)$ if and only if $f(x)=f(g(x))$ for every $x\in X$ and $g\in G$. In others words $\mathcal{T}(R)$ is the set of <u>G-invariant maps from X to L</u>.

It is interesting to examine the fuzzy continuity defined by these L-topologies. In the sequel $\bar{R}$ denote the order-relation generated by R, i.e. $\bar{R}=\{(x,y)\in X\times X/\text{there exists } x_1,\dots,x_n \text{ such that } xRx_1, x_1Rx_2,\dots,x_nRy\}\cup\{(x,x)\in X\times X/x\in X\}$.

PROPOSITION 5. Let (X,R) and (X',R') two graph, then a map $h:X\longrightarrow X'$ is L-continuous, with respect to $\mathcal{T}(R)$ and $\mathcal{T}(R')$, if and only if it is an increasing map with respect $\bar{R}$ and $\bar{R}'$.

Proof. Recall that h is L-continuous if and only if $h^{-1}(g)\in\mathcal{T}(R)$ for every $g\in\mathcal{T}(R')$, if and only if xRy implies $g(h(x))\leq g(h(y))$ for every $g\in\mathcal{T}(R')$, $x\in X$ and $y\in Y$. Now if h is increasing, from xRy it follows that $h(x)\bar{R}'h(y)$. Then either there exists $x_1,\dots,x_n\in X'$ such that $h(x)R'x_1$, $x_1R'x_2,\dots,$ $x_nR'h(y)$ or $h(x)=h(y)$. It follows that $g(h(x))\leq g(h(y))$ and this proves that h is L-continuous.

Conversely, assume h L-continuous and, for every $b\in X'$, let g_b be defined by setting

$$g_b(x)=\begin{cases}0 \text{ if } x\bar{R}'b\\ 1 \text{ otherwise.}\end{cases}$$

We will prove that $g_b\in\mathcal{T}(R')$, i.e. that $xR'y$ implies $g_b(x)\leq g_b(y)$. Now if $g_b(x)=0$ then $g_b(x)\leq g_b(y)$. If $g_b(x)=1$, then from $g_b(y)=0$, i.e. $y\bar{R}'b$, and $xR'y$ it follows that $x\bar{R}'b$, that is $g_b(x)=0$, absurd. It follows that $g_b(y)=1$ and therefore that $g_b(x) \leq g_b(y)$. This proves that $g_b \in \mathcal{T}(R')$. Now from the L-continuity of h it follows that xRy implies $g_b(h(x)) \leq g_b(h(y))$ for every $b \in X'$. By setting $b=h(y)$, we have that $g_{h(y)}(h(y))=0$, and $g_{h(y)}(h(x))=0$. In conclusion from xRy it follows that $h(x)\bar{R}'h(y)$. Moreover if $x\bar{R}y$, i.e. if there exists $x_1,\dots,x_n \in X$ such that xRx_1, $x_1Rx_2,\dots,x_nRy$ or $x=y$, it follows that $h(x)\bar{R}'h(x_1)$, $h(x_1)\bar{R}'h(x_2),\dots$, $h(x_n)\bar{R}'h(y)$ or $h(x)=h(y)$. This proves that $h(x)\bar{R}'h(y)$.

If R and R' are order-relations, then $R=\bar{R}$ and $R'=\bar{R}'$. Then from Proposition 5 it follows that h is L-continuous if and only if it is increasing. Then we can assert that the map H, defined by setting $H(R)=\mathcal{T}(R)$ and $H(h)=h$, is a functor from the category of the ordered sets into the category of the L-topological spaces.

If R is defined by (5) and R' is the identity relation, then $R=\bar{R}$ and $R'=\bar{R}'$ and $h:X \rightarrow X'$ is L-continuous if and only if $h(x)=h(g(x))$ for every $g \in G$. This proves that the L-continuous maps from X to X' coincide with the G-invariant maps.

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