TNF-MEASURE ON TNF-MEASURABLE SPACE

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Following the suggestion made by Klement[1] and Yu Yandong[2], a new concepte of TNF-measure on TNF-measurable space is given, T being any measurable trianglar norm and N any negation. Most of the results about TF-measures obtained in[1] are extended to the case of TNF-measures. Some other properties of TNF-measure are also discussed.

Keywords: Triangular norm(t-norm), Negation, TNF-σ-algebra TNF-measurable space, TNF-measure, TNNF-measure.

I. Introduction

In 1982, Klement[1] first introduced the notion of TF-σ-algebras and TF-measure. In 1985, Yu Yandong continuing Klement's work on TF-σ-algebras given an axiomatic theory of TNF- σ-algebras.

In the present paper, continuting Yu's work on TNF-σ-algebras, we further give a new concept of TNF-measure on TNF-measurable space, where the operations intersection and union of fuzzy sets are assumed to be measurable t-norms and their N-duals, N being any negation(cf., e.g. [1, 2].)

In Section II we give some characterizations of these TN-fuzzy measures and the extension theorem of TNF-measure.

Final we give a notion of TN-non-additive-fuzzy measure we would study its properties in other paper.

II. TNF-σ-algebras

Throughout this paper U will denote a nonempty set. The unit interval I=[0,1] will always be equipped with $\mathcal{B}$, the usual σ-algebra of Borel subsets of [0,1].

Now let us recall some basic definitions and mathematical facts about fuzzy sets and TNF-σ-algebras.

Definition2.1. A fuzzy set on U is a function $x:U \longrightarrow [0,1]$. As usual, the family of all fuzzy sets on U is denoded by $F=[0,1]^U$.

Definition 2.2.Let T be a t-norm, S an s-norm, and N a negation. Given any $x,y \in F$, $T(x,y)$, $S(x,y)$ and $N(x)$ are the fuzzy sets on U determined, respectively, by

$$T(x,y)(u) = T(x(u),y(u)),$$
$$S(x,y)(u) = S(x(u),y(u)),$$
$$N(x)(u) = N(x(u)),$$

for any $u \in U$.

Let T be a t-norm and N anegation. Then it is easy to see that there exists one and only one s-norm satisfying the identity:

(I) $N(T(x,y)) = S(N(x),N(y))$ for all $x,y \in F$.

The s-norm S is called the N-dual of T (cf.[1,2]).

Note that the idetity (I) is equivalent to the identity

(II) $N(S(x,y)) = T(N(x),N(y))$ for all $x,y \in F$.

this moment

$$S(x,y) = N(T(N(x),N(y))).$$

Definition 2.3. Let T be a measurable t-norm, N a negation and S the N-dual of T. A subfamily σ of F is called a fuzzy σ-algebra on U with respect to the t-norm T and the negation N, or TNF-σ-algebra for short, if and only if the following axioms hold:

Axiom 1. α constant $\Rightarrow \alpha \in \sigma$;

Axiom 2. $x \in \sigma \Rightarrow N(x) \in \sigma$;

Axiom 3. $(x_i)_{i \in \mathbb{N}} \in \sigma^{\mathbb{N}} \Rightarrow S_{i=1}^{\infty} x_i \in \sigma$.

If σ is a TNF-σ-algebra on U, the pair (U, σ) is called a TNF-measurable space.

Obviously, both Axiom 2 and Axiom 3 hold if and only if both axiom 2 and the following axiom 3' hold.

Axiom 3'. $(x_i)_{i \in \mathbb{N}} \in \sigma^{\mathbb{N}} \Rightarrow T_{i=1}^{\infty} x_i \in \sigma$.

Particularly, if N_0: $N_0(x) = 1-x$ for all $x \in \sigma$, in the case, then TN_0F- σ-algebra is a TF- σ-algebra as in [1].

Definition 2.4. Let (U, ξ) and (V, σ) be TNF-measurable space and f: $U \to V$ a mapping. The mapping f is called a TNF-measurable mapping from (U, ξ) into (V, σ) if and only if

$$f^{-1}(\sigma) = \{x \circ f \mid x \in \sigma\} \subset \xi.$$

Proposition 2.1. Let (V, σ) be a TNF-measurable space and f: $U \to V$ a mapping. Then $f^{-1}(\sigma)$ is a TNF- σ-algebra on U and is the smallst of all TNF- σ-algebras ξ on U making f a TNF-measurable mapping from (U, ξ) into (V, σ).

Definition 2.5. Given any σ-algebra $\mathcal{A}$ on U, $\zeta(\mathcal{A})$ denotes the family of all measurable functions from $(U, \mathcal{A})$ into $([0,1], \mathcal{B})$.

Definition 2.6. A TNF- σ-algebra σ on U is said to be generated if and only if there exists a σ-algebra $\mathcal{A}$ on U such that

$$\sigma = \zeta(\mathcal{A}).$$

Proposition 2.2. Let $\mathcal{A}$ be a σ-algebra on U, and T a measurable t-norm. Then for each negation N, $\zeta(\mathcal{A})$ is a TNF- σ-algebra on U.

Definition 2.7. Given any family $\mathcal{A}$ of fuzzy sets on U, $\sigma_{TN}(\mathcal{A})$ denotes the smallest TNF- σ-algebra on U containing $\mathcal{A}$ as a subfamily.

Theorem 2.3. Let T be a continuous t-norm and N a negation Let V be a nonempty set, σ a generated TNF- σ-algebra on V and $f: U \to V$ a mapping. Then $f^{-1}(\sigma)$ is a generated TNF- σ-algebra on U.

III. TNF-measures on TNF-measurable space

In this section, we first give a definition of finite TN-fuzzy measure on TNF-measurable space. Then we show that each finite TN-fuzzy measure is a T_0N-fuzzy measure. Next, we study the relationship between TN-fuzzy measure and integrals. Finally, we give a definition of TN-non-additive-fuzzy measure on TNF-measurable space.

Definition 3.1. Let T be a measurable t-norm, N a negation S the N-dual of T, and (U, σ) a TNF-measurable space. A mapping $m: \sigma \longrightarrow R = [0, +\infty)$ is called a finite TN-fuzzy measure or TNF-measure for short, if and only if fulfills the following properties:

(TNFM1). $m(0) = 0$;

(TNFM2). $m(T(x,y)) + m(S(x,y)) = m(x) + m(y)$ for all $x, y \in \sigma$;

(TNFM3). $\forall (x_n)_{n \in \mathbb{N}} \subset \sigma$, $(x_n)_{n \in \mathbb{N}} \uparrow x$, $x \in \sigma \Rightarrow (m(x_n))_{n \in \mathbb{N}} \uparrow m(x)$

If m is a TNF-measure on TNF-measurable space (U, σ), the (U, σ, m) is called a TNF-measure space.

Note that in (TNFM3) it is necessare to require $x = \sup x_n \in \sigma$ explicitly. From [3] we know that in general the supremum of a sequence of elements in σ need not be an element of σ.

Of couse, the finite T-fuzzy measures considered in [1] are now finite TN_0F-measures in this more general context, where $N_0 : N_0(x) = 1 - x$ for all $x \in \sigma$.

Theorem 3.1. Let T be a measurable t-norm, N a negation and σ both a TN-fuzzy and T_0N-fuzzy σ-algebra on U. Then each finite TNF-measure on (U, σ) is a T_0NF-measure.

Theorem 3.2. Let $(U, \mathcal{A})$ be a measurable space, T a measurable t-norm, N a negation, S a N-dual of T, and P a finite measure on $(U, \mathcal{A})$ such that $P(U) > 0$ and define

$$m: \mathcal{F}(\mathcal{A}) \longrightarrow R = [0, +\infty)$$

by $m(x) = \int x dP$ for all $x \in \mathcal{J}(\mathcal{A})$, then m is a finite TNF-measure if and only if the pair (T,S) fulfills the functional equation

$$T(x,y)+S(x,y) = x+y \tag{3.1}$$

Proposition 3.3 Let T_s be a measurable t-norm, N a negation S a N-dual of T_s such that (3.1) holds, $(U, \sigma_s = \zeta_{TN}(\mathcal{A}))$ a T_sNF-measurable space, $0 < s < +\infty$, m a T_sNF-measure on (U, σ_s). Then the following assertions are equivalent:

(i).m is an integral [i.e., there exists a finite measure P on $(U, \mathcal{A})$ such that for all $x \in \sigma_s$ $m(x) = \int x dP$ holds];

(ii).m is additive [i.e., for all $x, y \in \sigma_s : x+y \in \sigma_s \Rightarrow$ $m(x+y) = m(x)+m(y)$];

(iii).m is null-continuous from above [i.e., when $x_n \downarrow 0$, $x_n \in \sigma_s$, $n=1,2,\cdots$, $\Rightarrow m(x_n)_{n\in N} \downarrow 0$].

Proof. Since (i)$\Leftrightarrow$(ii) and (i)$\Rightarrow$(iii) by the integral convergence theorem (cf.,[1]). We only have to show the validity of (iii)$\Rightarrow$(ii). $\forall x \in \sigma_s$, $y \in \sigma_s$, $x+y \in \sigma_s$, Let $x_0 = x$ $y_0 = y, \cdots, x_{n+1} = S_s(x_n, y_n)$, $y_{n+1} = T_s(x_n, y_n)$, from [3] $\Rightarrow x_n \downarrow$ $x+y$, $y_n \downarrow 0$. Thus

$$\begin{aligned} m(x)+m(y) &= m(T_s(x,y))+m(S_s(x,y)) \\ &= m(x_1)+m(y_1) \\ &= m(T_s(x_1,y_1))+m(S_s(x_1,y_1)) \\ &= m(x_2)+m(y_2) = \cdots = m(x_n)+m(y_n), \ \forall n \in N, \end{aligned}$$

by the null-continuouty from above of m $\Rightarrow \lim_{n\to\infty} m(y_n) = 0$, so that

$$m(x)+m(y) = \lim_{n\to\infty} (m(x_n)+m(y_n)) = m(x+y).$$

Theorem 3.4. Let T be a measurable t-norm, if T fulfills $T(x,y)=0$, when $x, y \in F$, $x \vee y < 1$, $1-a \leqslant x+y \leqslant 1+a$ and $1-a \leqslant N(x)+N(y) \leqslant 1+a$ for a given $a \in (0,1]$, S a N-dual of T. Then the family of all finite TNF-measures on (U, σ) to be a family of zero measure.

Proof. Since, when $N(x)\wedge N(y)>0$, $1-a\leqslant x+y\leqslant 1+a$ and $1-a\leqslant N(x)+N(y)\leqslant 1+a$, S fulfills:

$$S(x,y) = N(T(N(x),N(y))) = 1.$$

When $\frac{1}{2}(1-a)\leqslant b\leqslant \frac{1}{2}(1+a)$,

$$m(b)=\frac{1}{2}(m(T(b,b))+m(S(b,b)))= \frac{1}{2}m(1).$$

Choose $x\in\left[\frac{1}{2}(1+a),\ \frac{1}{2}(1+3a)\wedge 1\right]$, owing to $1\leqslant \frac{1}{2}(1-a)+x\leqslant 1+a$, therefore

$$m(x)+m(\tfrac{1}{2}(1-a))=m(T(x,\tfrac{1}{2}(1-a))+m(S(x,\tfrac{1}{2}(1-a))=m(1),$$

$$m(x)=m(1)-m(\tfrac{1}{2}(1-a))=\tfrac{1}{2}m(1),$$

again choose $x\in\left(\frac{1}{2}(1-3a)\vee 0, \frac{1}{2}(1-a)\right]$, from continuty from below of m and $1\leqslant x+\frac{1}{2}(1+3a)\wedge 1\leqslant 1+a$,

$$m(\tfrac{1}{2}(1+3a)\wedge 1)= \tfrac{1}{2}m(1),$$

$$m(x)=m(T(x,\tfrac{1}{2}(1+3a)\wedge 1))+m(S(x,\tfrac{1}{2}(1+3a)\wedge 1))-$$
$$m(\tfrac{1}{2}(1+3a)\wedge 1) =m(1)- \tfrac{1}{2}m(1)=\tfrac{1}{2}m(1).$$

Since $a>0 \Longrightarrow \exists n\in\mathbb{N}$ such that $\frac{1}{2}(1+(2n+1)a)>1$, $\Longrightarrow$ when $x\in(0,1)$,

$$m(x)=\tfrac{1}{2}m(1).$$

By the continuty from below of m, we obtain

$$m(1)=\tfrac{1}{2}m(1)$$

$$\Longrightarrow m(1)=0 \Longrightarrow m=0.$$

The proof is completed.

Proposition 3.5. Let (U, σ) be a TNF-measurable space, m_1 a TNF-measure on $\{\alpha \mid \alpha\in[0,1]\}$. Any taken $u_\circ\in U$, $\forall\, x\in\sigma$, defining:

$$m(x) = \begin{cases} m_1(x(u_\circ)) & x(u_\circ)>0 \\ 0 & x(u_\circ)=0 \end{cases}$$

then m is a TNF-measure on (U, σ).

Theorem 3.6 Let (U, ξ) and (V, σ) be TNF-measurable spaces and $f: (U,\xi) \to (V, \sigma)$ a TNF-measurable mapping. If m is a TNF-measure on (U, ξ), the

$$\overline{m}: \sigma \to R=[0,\infty)$$
$$z \mapsto m(f^{-1}(z))=m(z\circ f)$$

a TNF-measure on (V, σ).

Proof. (i) $\overline{m}(0)=m(f^{-1}(0))=m(0\circ f)=m(0)=0$.

Since $\forall u\in U$ $f^{-1}(T(z,w))(u)=((T(z,w))\circ f)(u)$

$$=T(z,w)\circ f(u)=T(z(f(u)),w(f(u)))$$
$$=T((z\circ f)(u),(w\circ f)(u))$$
$$=T(f^{-1}(z)(u),f^{-1}(w)(u))$$

i.e., $T(z,w)\circ f= T(z\circ f,w\circ f)$, where $z,w\in\sigma$.

Thus (ii) $\forall z,w\in\sigma$,

$$\overline{m}(T(z,w))+\overline{m}(S(z,w))$$
$$= m(f^{-1}(T(z,w)))+m(f^{-1}(S(z,w)))$$
$$= m(T(z,w)\circ f)+m(S(z,w)\circ f)$$
$$= m(T(z\circ f,w\circ f))+m(S(z\circ f,w\circ f))$$
$$= m(z\circ f)+m(w\circ f) = \overline{m}(z)+\overline{m}(w)$$

(iii) $\forall z_n\in\sigma$, $n\in\mathbb{N}$, $(z_n)_{n\in\mathbb{N}}\uparrow z$, $z\in\sigma \Longrightarrow$
$(\overline{m}(z_n))_{n\in\mathbb{N}}= (m(z_n\circ f))_{n\in\mathbb{N}}\uparrow m(z\circ f)=\overline{m}(z)$.

The theorem is proved.

Corollary 3.7. Let (U,ξ) and (V,σ) be TNF-measurable spaces and $f: U\longrightarrow V$ a mapping. Assume tnat T is continuous and assume that both ξ and σ are generated and $\xi=\mathcal{J}(\mathcal{A})$, $\sigma=\mathcal{J}(\mathcal{D})$, and f is a measurable mapping from $(U,\mathcal{A})$ into $(V,\mathcal{D})$, m a TNF-measure on (U,ξ), tnen

$$\overline{m}: \sigma\longrightarrow [0,\infty)$$
$$z\longmapsto m(f^{-1}(z))=m(z\circ f)$$

is a TNF-measure on (V,σ).

Definition 3.2. Let T be a measurable t-norm, N a negation, S the N-dual of T and (U,σ) a TNF-measurable space. A mapping $m: \sigma\longrightarrow R=[0,\infty)$ is called a finite TN-non-additive-fuzzy measure or TNNF-measure for short, if and only if it fulfills the properties:

(TNNFM1) $m(0)=0$;

(TNNFM2) $\forall x,y\in\sigma$, $x\leqslant y\Rightarrow m(x)\leqslant m(y)$ (Monotonicity);

(TNNFM3) $\forall\{x_n\}\subset\sigma$, $x_n\uparrow x$, $x\in\sigma \Longrightarrow m(S_{n=1}^{\infty})=\lim_{n\to\infty}m(x_n)$;

(TNNFM4) $\forall\{x_n\}\subset\sigma$, $x_n\downarrow x$, $x\in\sigma \Longrightarrow m(T_{n=1}^{\infty})=\lim_{n\to\infty}m(x_n)$.

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