The Fuzzy Integral and the Convergence Theorems

Qiao Zhong

Hebei Institute of Architectural Engineering

Zhangjiakou, Hebei, China

Abstract

In this paper, some properties of the fuzzy integrals on the fuzzy sets are discussed, and some necessary and sufficient conditions for the convergence of a sequence of the fuzzy integrals are given.

*1. The Fuzzy Integral on the Fuzzy Set

In this paper, we shall further discuss the fuzzy integrals on the fuzzy sets introduced in [3,5,6], some new properties of the fuzzy integrals on the fuzzy sets will be discussed, and some convergence theorems of a sequence of the fuzzy integrals will be proved.

All concepts and signs not defined in this paper may be found in [1,2,3,4,5,6].

Throughout this paper, let X be a classical nonempty set, $\mathcal{F}(X)=\{\underset{\sim}{A};\ \underset{\sim}{A}: X\rightarrow[0,1]\}$ be the class of all fuzzy subsets of X, $\underset{\sim}{\mathcal{F}}\subset\mathcal{F}(X)$ be a fuzzy σ-algebra of fuzzy sets, $\underset{\sim}{\mu}: \underset{\sim}{\mathcal{F}}\rightarrow[0,\infty]$ be a fuzzy measure on $(X, \underset{\sim}{\mathcal{F}})$, and

$\underset{\sim}{M}=\{f;\ f: X\rightarrow(-\infty,\infty),\ F_\alpha=\{x;\ f(x)\geqslant\alpha\}\in\underset{\sim}{\mathcal{F}},\ \forall\ \alpha\in[-\infty,\infty]\}$ be the set of all fuzzy measurable functions on $(X,\underset{\sim}{\mathcal{F}})$, $\underset{\sim}{M}^+=\{f; f\in\underset{\sim}{M}, f\geqslant 0\}$. We make the following conventions: $\sup\limits_{t\in\phi}\{a_t;\ a_t\in[0,\infty]\}=0$,

$\inf\limits_{t\in\phi}\{a_t;\ a_t\in[0,\infty]\}=\infty$.

Definition1.1 Let $A \in \mathcal{F}$ with $\mu(A) < \infty$. μ is called pseudo-null-subtractive with respect to A, if for any $E \in A \cap \mathcal{F}$, we have $\mu(E \cap B) = \mu(E)$, whenever $B \in \mathcal{F}$ and $\mu(A \cap B) = \mu(A)$.

Definition1.2 μ is called null-subtractive (resp. null-additive), if we have $\mu(A \cap B^c) = \mu(A)$ (resp. $\mu(A \cup B) = \mu(A)$), whenever $A, B \in \mathcal{F}$ and $\mu(B) = 0$.

Definition1.3 Let $\{f_n\} \subset M$, $f \in M$, $A \in \mathcal{F}$, $D = \{x; f_n(x) \to f(x)\}$.
(1) If $A \subset D$, then we say $\{f_n\}$ converges to f everywhere on A, and denote it by $f_n \xrightarrow{e.} f$ on A ;
(2) If $\mu(A \cap \{|f_n - f| \geq \varepsilon\}) \to 0$ for any given $\varepsilon > 0$, then we say $\{f_n\}$ converges in fuzzy measure μ to f on A, and denote it by $f_n \xrightarrow{\mu} f$ on A.

Lemma1.4 Let $f \in M^+$, $F_\alpha = \{x; f(x) \geq \alpha\}$, $F_{\bar{\alpha}} = \{x; f(x) > \alpha\}$, then $\lim\limits_{\beta \to \alpha - 0} F_\beta = \lim\limits_{\beta \to \alpha - 0} F_{\bar{\beta}} = F_\alpha \supset F_{\bar{\alpha}} = \lim\limits_{\beta \to \alpha + 0} F_\beta = \lim\limits_{\beta \to \alpha + 0} F_{\bar{\beta}}$.

Theorem1.5 [1,2] Let $\mathcal{B}$ be a classical σ-algebra of X, μ^* be a fuzzy measure on $(X, \mathcal{B})$. Whenever $D \in \mathcal{B}$ and f_n, f are $\mathcal{B}$-measurable functions and $f_n \xrightarrow{\mu^*} f$ on D, then $\int_D f_n d\mu^* \to \int_D f d\mu^*$, if and only if μ^* is autocontinuous.

Theorem1.6 [1] Let $\mathcal{B}$ be a classical σ-algebra of X, μ^* be a fuzzy measure on $(X, \mathcal{B})$. Whenever $D \in \mathcal{B}$ and f_n, f are $\mathcal{B}$-measurable functions and $f_n \xrightarrow{a.e.} f$ on D, and there exist n_0 and a constant $c \leq \int_D f d\mu^*$, such that $\mu^*(\{\sup\limits_{i \geq n_0} f_i > c\} \cap D) < \infty$, then $\int_D f_n d\mu^* \to \int_D f d\mu^*$, if and only if μ^* is null-additive.

Definition1.7 Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space,

$\underset{\sim}{A} \in \mathcal{F}$, $f \in \underset{\sim}{M}^{+}$. The fuzzy integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha)]$

where $F_\alpha = \{x;\ f(x) \geq \alpha\}$, $\alpha \in [0, \infty]$.

Proposition1.8 $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{\alpha \in [0,\infty)} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha)] = \sup_{\alpha \in (0,\infty)} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha)]$.

Proposition1.9 $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{\alpha \in [0,\infty)} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})] = \sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})]$.

where $F_{\bar{\alpha}} = \{x;\ f(x) > \alpha\}$, $\alpha \in [0, \infty]$.

Proof. We only prove $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})]$.

By using the monotonicity of $\underset{\sim}{\mu}$, we have $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha) \geq \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})$ for any $\alpha \in [0, \infty]$, and therefore, $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \geq \sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})]$. We assume that $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} > \sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})] = b$, then there exists $\varepsilon > 0$, such that $\sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha)] > b+\varepsilon$, and therefore, there exists α_0, such that $\alpha_0 \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\alpha_0}) > b+\varepsilon$, namely, $\alpha_0 > b+\varepsilon$ and $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\alpha_0}) > b+\varepsilon$. We have $\underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\overline{b+\varepsilon}}) \geq \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\alpha_0}) > b+\varepsilon$. Therefore, $\sup_{\alpha \in [0,\infty]} [\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\bar{\alpha}})] \geq (b+\varepsilon) \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_{\overline{b+\varepsilon}}) = b+\varepsilon > b$. It is a contradiction. The proof of the proposition is complete.

Proposition1.10
$$\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{E \in \mathcal{B}(f)} [(\inf_{x \in E} f(x)) \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap E)]$$
$$= \sup_{\underset{\sim}{E} \in \mathcal{F}} [(\inf_{\underset{\sim}{E}(x) > 0} f(x)) \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap \underset{\sim}{E})]$$

where $\mathcal{B}(f)$ is the classical σ-algebra generated by f . (Obviously, $\mathcal{B}(f) \subset \mathcal{F}$)

Proof. First, for any given $\alpha \in [0, \infty)$, we have $\inf_{x \in F_\alpha} f(x) \geq \alpha$. Since $F_\alpha \in \mathcal{B}(f)$, then $[\alpha \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha)] \leq \sup_{E \in \mathcal{B}(f)} [(\inf_{x \in E} f(x)) \wedge \underset{\sim}{\mu}(\underset{\sim}{A} \cap E)]$.

Therefore, $\int_{A} f d\mu \leq \sup_{E\in\mathcal{B}(f)} [(\inf_{x\in E} f(x))\wedge\mu(A\cap E)]$.

Furthermore, since $\mathcal{B}(f)\subset\mathcal{F}$, then

$$\sup_{E\in\mathcal{B}(f)} [(\inf_{x\in E} f(x))\wedge\mu(A\cap E)] \leq \sup_{E\in\mathcal{F}} [(\inf_{E(x)>0} f(x))\wedge\mu(A\cap E)] .$$

Finally, for any given $E\in\mathcal{F}$, if we take $\alpha'=\inf_{E(x)>0} f(x)$, then $E\subset F_{\alpha'}$, and therefore, $\mu(A\cap E)\leq\mu(A\cap F_{\alpha'})$ and $(\inf_{E(x)>0} f(x))\wedge\mu(A\cap E)\leq\alpha'\wedge\mu(A\cap F_{\alpha'})\leq\int_{A} f d\mu$. It follows that $\sup_{E\in\mathcal{F}} [(\inf_{E(x)>0} f(x))\wedge\mu(A\cap E)]\leq\int_{A} f d\mu$. The proof of the proposition is complete.

Theorem 1.11 The fuzzy integrals on the fuzzy sets satisfy the following properties:

(1) If $\mu(A) = 0$, then $\int_{A} f d\mu = 0$;

(2) If $\int_{A} f d\mu = 0$, then $\mu(A\cap F_{\bar{0}}) = 0$;

(3) If $f_1\leq f_2$, then $\int_{A} f_1 d\mu\leq\int_{A} f_2 d\mu$;

(4) If $A\subset B$, then $\int_{A} f d\mu\leq\int_{B} f d\mu$;

(5) $\forall a\in[0,\infty)$, $\int_{A} a d\mu = a\wedge\mu(A)$;

(6) $\int_{A}(f_1\vee f_2)d\mu\geq\int_{A} f_1 d\mu\vee\int_{A} f_2 d\mu$;

(7) $\int_{A}(f_1\wedge f_2)d\mu\leq\int_{A} f_1 d\mu\wedge\int_{A} f_2 d\mu$;

(8) $\int_{A\cup B} f d\mu\geq\int_{A} f d\mu\vee\int_{B} f d\mu$;

(9) $\int_{A\cap B} f d\mu\leq\int_{A} f d\mu\wedge\int_{B} f d\mu$;

(10) $\int_{A}(f+a)d\mu\leq\int_{A} f d\mu+\int_{A} a d\mu$, $a\in[0,\infty)$, $f\in M^{+}$;

(11) $\forall a\in[0,\infty)$, if $|f_1-f_2|\leq a$, then

$$\left|\int_{A} f_1 d\mu-\int_{A} f_2 d\mu\right|\leq a.$$

Proof. We only prove (2), (3), (5), (10), (11).

(2) Let $\int_A f d\mu = 0$ and $\mu(A\cap F_{\bar{0}}) = c>0$, by using Lemma1.4, we have $A\cap F_{\frac{1}{n}} \nearrow A\cap F_{\bar{0}}$ as $n\to\infty$. It follows from the continuity from below of μ that $\mu(A\cap F_{\frac{1}{n}})\to\mu(A\cap F_{\bar{0}}) = c$. Therefore, there exists n_0 such that $\mu(A\cap F_{\frac{1}{n_0}}) \geq \frac{c}{2}$. We have

$\int_A f d\mu = \sup_{\alpha\in[0,\infty)}[\alpha\wedge\mu(A\cap F_\alpha)] \geq \frac{1}{n_0}\wedge\mu(A\cap F_{\frac{1}{n_0}}) \geq \frac{1}{n_0}\wedge\frac{c}{2}>0$. It is a contradiction.

(3) Let $f_1\leq f_2$, $F^k_\alpha=\{x;\ f_k(x)\geq\alpha\}$, $k=1,2$. Since $\mu(A\cap F^1_\alpha)\leq\mu(A\cap F^2_\alpha)$, then $\int_A f_1 d\mu\leq\int_A f_2 d\mu$.

(5) Since
$$F_\alpha=\{x;\ a\geq\alpha\}=\begin{cases}\phi & \text{if } a<\alpha\\ X & \text{if } a\geq\alpha,\end{cases}$$

then
$$\int_A a d\mu = \sup_{0\leq\alpha\leq a}[\alpha\wedge\mu(A\cap F_\alpha)]\vee\sup_{a<\alpha\leq\infty}[\alpha\wedge\mu(A\cap F_\alpha)]$$
$$=\sup_{0\leq\alpha\leq a}[\alpha\wedge\mu(A)]\vee 0 = a\wedge\mu(A).$$

(10) By using proposition1.10, we have
$$\int_A(f+a)d\mu = \sup_{E\in\mathcal{F},\ E(x)>0}[(\inf(f(x)+a))\wedge\mu(A\cap E)]$$
$$\leq\sup_{E\in\mathcal{F},\ E(x)>0}[(\inf f(x))\wedge\mu(A\cap E)+(a\wedge\mu(A\cap E))]$$
$$\leq\int_A f d\mu+\int_A a d\mu.$$

(11) Let $|f_1-f_2|\leq a$, since $f_1\leq f_2+a$, then
$$\int_A f_1 d\mu\leq\int_A(f_2+a)d\mu\leq\int_A f_2 d\mu + a\wedge\mu(A)\leq\int_A f_2 d\mu + a,$$
therefore, $\int_A f_1 d\mu\leq\int_A f_2 d\mu + a$. Analogously, we can prove $\int_A f_2 d\mu\leq\int_A f_1 d\mu + a$. Thus, we have $|\int_A f_1 d\mu - \int_A f_2 d\mu|\leq a$.

Theorem1.12 $\int_A f d\mu<\infty$, if and only if there exists $\alpha_0\in[0,\infty)$, such that $\mu(A\cap F_{\alpha_0})<\infty$.

Proof. If there exists $\alpha_0\in[0,\infty)$, such that $\mu(A\cap F_{\alpha_0})=a<\infty$, then $\mu(A\cap F_\alpha)\le\mu(A\cap F_{\alpha_0})=a$, for any $\alpha>\alpha_0$. Consequently,

$$\int_A f d\mu=\sup_{\alpha\in[0,\alpha_0]}[\alpha\wedge\mu(A\cap F_\alpha)]\vee\sup_{\alpha\in[\alpha_0,\infty)}[\alpha\wedge\mu(A\cap F_\alpha)]\le\alpha_0\vee a<\infty.$$

Conversely, if for any $\alpha\in[0,\infty)$, $\mu(A\cap F_\alpha)=\infty$, then

$$\int_A f d\mu=\sup_{\alpha\in[0,\infty)}[\alpha\wedge\mu(A\cap F_\alpha)]=\sup_{\alpha\in[0,\infty)}\alpha=\infty.$$

Theorem 1.13 Let $A\in\mathcal{F}$, $\alpha\in[0,\infty)$, then

(1) $\int_A f d\mu\ge\alpha\Leftrightarrow\forall\beta\in[0,\alpha),\ \mu(A\cap F_\beta)\ge\alpha\Leftarrow\mu(A\cap F_\alpha)\ge\alpha$;

$\int_A f d\mu<\alpha\Leftrightarrow\exists\beta\in[0,\alpha)$, such that $\mu(A\cap F_\beta)<\alpha\Longrightarrow\mu(A\cap F_\alpha)<\alpha$

$\Rightarrow\mu(A\cap F_{\underline{\alpha}})<\alpha$;

(2) $\int_A f d\mu\le\alpha\Longleftrightarrow\mu(A\cap F_{\underline{\alpha}})\le\alpha\Longleftarrow\mu(A\cap F_\alpha)\le\alpha$;

$\int_A f d\mu>\alpha\Leftrightarrow\mu(A\cap F_{\underline{\alpha}})>\alpha\Rightarrow\mu(A\cap F_\alpha)>\alpha$;

(3) $\int_A f d\mu=\alpha\Leftrightarrow\forall\beta\in[0,\alpha),\ \mu(A\cap F_\beta)\ge\alpha\ge\mu(A\cap F_{\underline{\alpha}})$;

Particularly, if $\mu(A)<\infty$, then

$$\int_A f d\mu=\alpha\Leftrightarrow\mu(A\cap F_\alpha)\ge\alpha\ge\mu(A\cap F_{\underline{\alpha}}).$$

Proof. (1) It is sufficient to consider the case $\alpha\in(0,\infty)$. If $\mu(A\cap F_\beta)\ge\alpha$ for any $\beta<\alpha$, then

$$\int_A f d\mu\ge\sup_{\beta\in[0,\alpha)}[\beta\wedge\mu(A\cap F_\beta)]\ge\sup_{\beta\in[0,\alpha)}[\beta\wedge\alpha]=\sup_{\beta\in[0,\alpha)}\beta=\alpha.$$

On the other hand, if there exists $\beta<\alpha$, such that $\mu(A\cap F_\beta)<\alpha$, then $\mu(A\cap F_r)\le\mu(A\cap F_\beta)$ whenever $r\ge\beta$, thus we have

$$\int_A f d\mu=\sup_{r\in[0,\beta)}[r\wedge\mu(A\cap F_r)]\vee\sup_{r\in[\beta,\infty)}[r\wedge\mu(A\cap F_r)]\le\beta\vee\mu(A\cap F_\beta)<\alpha.$$

The equivalent relations are proved.

(2) If $\int_A f d\mu > \alpha$, then there exists $\alpha_0 > \alpha$, such that $\int_A f d\mu \geq \alpha_0$. It follows, by using (1), that $\mu(A \cap F_\beta) \geq \alpha_0$, whenever $\beta < \alpha_0$. Particularly, if we take $\beta = \frac{\alpha_0 + \alpha}{2}$, then $\alpha < \beta < \alpha_0$, and therefore, $\mu(A \cap F_{\bar{\alpha}}) \geq \mu(A \cap F_\beta) \geq \alpha_0 > \alpha$.

On the other hand, let $\mu(A \cap F_{\bar{\alpha}}) > \alpha$, if $\beta_n \searrow \alpha$, it follows from Lemma1.4 that $A \cap F_{\beta_n} \nearrow A \cap F_{\bar{\alpha}}$. By the continuity from below of μ, we have $\mu(A \cap F_{\beta_n}) \to \mu(A \cap F_{\bar{\alpha}})$, and therefore, there exists $\alpha_0 > \alpha$, such that $\mu(A \cap F_{\alpha_0}) > \alpha$. Thus, we have

$$\int_A f d\mu \geq \alpha_0 \wedge \mu(A \cap F_{\alpha_0}) > \alpha .$$

The equivalent relations are proved.

By using (1) and (2) and Lemma1.4 and the continuity of μ, we can obtain results given in (3).

Proposition1.14 Let μ be null-subtractive (resp. μ be pseudo-null-subtractive with respect to A). For any $A, B \in \mathcal{F}$, we have $\int_{A \cap B^c} f d\mu = \int_A f d\mu$ whenever $\mu(B) = 0$ (resp. $\mu(A \cap B^c) = \mu(A) < \infty$).

Proposition1.15 If μ is null-additive, then for any A, $B \in \mathcal{F}$, $f \in M^+$, we have $\int_{A \cup B} f d\mu = \int_A f d\mu$, whenever $\mu(B) = 0$.

In the following, we shall introduce the concept of F-mean convergence of a sequence of fuzzy measurable functions, and we shall show that this concept is equivalent to convergence in fuzzy measure.

Definition1.16 Let $\{f_n\} \subset M$, $f \in M$, $A \in \mathcal{F}$. $\{f_n\}$ is said to F-mean converge to f on A, if

$$\lim_{n \to \infty} \int_A |f_n - f| d\mu = 0.$$

Theorem1.17 F-mean convergence is equivalent to convergence in fuzzy measure.

Proof. If $f_n \xrightarrow{\mu} f$ on $\underset{\sim}{A}$, then for any given $\varepsilon > 0$, there exists n_ε, such that $\mu(\underset{\sim}{A} \cap \{|f_n - f| \geq \frac{\varepsilon}{2}\}) < \varepsilon$ as $n \geq n_\varepsilon$. It follows, by using Theorem1.13(1), that $\int_{\underset{\sim}{A}} |f_n - f| d\mu < \varepsilon$ as $n \geq n_\varepsilon$. Namely, $\{f_n\}$ F-mean converges to f.

Conversely, if $\{f_n\}$ does not converge in fuzzy measure μ to f on $\underset{\sim}{A}$, then there exist $\varepsilon > 0$, $\delta > 0$, and a sequence $\{n_i\}$, such that $\mu(\underset{\sim}{A} \cap \{|f_{n_i} - f| \geq \varepsilon\}) > \delta$ for every n_i. It follows that $\int_{\underset{\sim}{A}} |f_{n_i} - f| d\mu \geq \varepsilon \wedge \mu(\underset{\sim}{A} \cap \{|f_{n_i} - f| \geq \varepsilon\}) \geq \varepsilon \wedge \delta > 0$, for every n_i. That is to say, $\{f_n\}$ does not F-mean converge to f.

*2. Convergence Theorems

Qiao [3,5,6] proved some convergence theorems of a sequence of the fuzzy integrals on the fuzzy sets. In this section, we shall give some necessary and sufficient conditions for the convergence of a sequence of the fuzzy integrals on the fuzzy sets.

Definition2.1 Let $\{f_n\} \subset \underset{\sim}{M}^+$, $f \in \underset{\sim}{M}^+$, $\mathscr{B}(\underset{\sim}{M}^+)$ be the classical σ-algebra generated by all functions in $\underset{\sim}{M}^+$. For any given $\underset{\sim}{A} \in \mathscr{F}$, we define $\mu^*(E) \triangleq \mu(\underset{\sim}{A} \cap E)$, for any $E \in \mathscr{B}(\underset{\sim}{M}^+)$. Obviously, $\mathscr{B}(\underset{\sim}{M}^+) \subset \mathscr{F}$, μ^* is a fuzzy measure on $(X, \mathscr{B}(\underset{\sim}{M}^+))$, we call μ^* a fuzzy measure induced by μ and $\underset{\sim}{A}$.

Theorem2.2 (Transformation Theorem) Let $(X, \mathscr{F}, \mu)$ be a fuzzy measure space, $(X, \mathcal{B}(M^+), \mu^*)$ be the fuzzy measure space induced by μ and A $(A \in \mathcal{F})$, then

$$\int_{A \cap D} f d\mu = \int_D f d\mu^* \text{ , whenever } D \in \mathcal{B}(M^+).$$

Theorem2.3 For any given $A \in \mathcal{F}$, whenever $\{f_n, f\} \subset M^+$ and $f_n \xrightarrow{\mu} f$ on A, then $\int_A f_n d\mu \to \int_A f d\mu$, if and only if μ^* is autocontinuous, where μ^* is the fuzzy measure induced by μ and A.

Proof. Necessity: For any $D \in \mathcal{B}(M^+)$ and $\{f_n, f\} \subset M^+$, if $f_n \xrightarrow{\mu^*} f$ on D, then $f_n \chi_D \xrightarrow{\mu^*} f \chi_D$ on X, and therefore, $f_n \chi_D \xrightarrow{\mu} f \chi_D$ on A. By hypothesis of the theorem, we have $\int_A f_n \chi_D d\mu \to \int_A f \chi_D d\mu$. It follows from Theorem2.2 that $\int_X f_n \chi_D d\mu^* \to \int_X f \chi_D d\mu^*$, namely, $\int_D f_n d\mu^* \to \int_D f d\mu^*$. It follows, by using Theorem1.5, that μ^* is autocontinuous.

Sufficiency: If $f_n \xrightarrow{\mu} f$ on A, and μ^* is autocontinuous, then $f_n \xrightarrow{\mu^*} f$ on X. By using Theorem1.5 and Theorem2.2, we have $\int_A f_n d\mu = \int_X f_n d\mu^* \longrightarrow \int_X f d\mu^* = \int_A f d\mu$.

The proof of the theorem is complete.

By using Theorem1.17 and Theorem2.3, we can give the following statement:

Theorem2.4 For any given $A \in \mathcal{F}$, whenever $\{f_n, f\} \subset M^+$ and $\{f_n\}$ F-mean converges to f on A, then $\int_A f_n d\mu \to \int_A f d\mu$, if and only if μ^* is autocontinuous, where μ^* is the fuzzy measure induced by μ and A.

Theorem2.5 Let $\{f_n, f\} \subset M^+$, $A \in \mathcal{F}$, if $f_n \nearrow f$ on A, then $\int_A f_n d\mu \nearrow \int_A f d\mu$.

Theorem2.6 Let $\{f_n, f\} \subset M^+$, $A \in \mathcal{F}$, if $f_n \searrow f$ on A, and there exist n_0 and a constant $c \leq \int_A f d\mu$ $(0 \leq c)$, such that $\mu(A \cap F_c^{n_0}) < \infty$, then $\int_A f_n d\mu \searrow \int_A f d\mu$.

[5,6] gave the proofs of Theorem2.5 and Theorem2.6.

Theorem2.7 For any given $A \in \mathcal{F}$, whenever $D \in \mathcal{F}(M^+)$, $\{f_n, f\} \subset M^+$ and $f_n \xrightarrow{a.e.} f$ on D (with respect to μ^*), and there exist n_0 and a constant $c \leq \int_{A \cap D} f d\mu$ $(0 \leq c)$, such that $\mu(\{\sup_{i \geq n_0} f_i > c\} \cap A \cap D) < \infty$, then $\int_{A \cap D} f_n d\mu \to \int_{A \cap D} f d\mu$, if and only if μ^* is null-additive.

Proof. By using Theorem1.6 and Theorem2.2, it is not difficult to prove this conclusion.

Note: Definition2.1 and Theorem2.2 given in this paper are aroused by Professor Wang Zhenyuan.

The author would like to thank Professor Wang Zhenyuan for his many valuable helps.

References

[1] Wang Zhenyuan, Asymptotic structural characteristics of fuzzy measure and their applications, Fuzzy Sets and Systems, 16(1985), 277-290.

[2] Wang Zhenyuan, The autocontinuity of set function and the fuzzy integral, J. Math. Anal. Appl.99(1984), 195-218.

[3] Qiao Zhong, The fuzzy integral on the fuzzy set, Journal of Hebei Institute of Architectural Engineering (in Chinese) ,2(1986).

[4] Qiao Zhong, Riesz's theorem and Lebesgue's theorem on the fuzzy measure space, BUSEFAL, 29(1987), 33-41.

[5] Qiao Zhong, The structural characteristics of the fuzzy measure on the fuzzy σ-algebra and their applications, Journal of Hebei Institute of Architectural Engineering,(in Chinese), 1(1987).

[6] Qiao Zhong, Convergence theorems of the sequence of the fuzzy integrals on the fuzzy sets, (in Chinese), May 1986. (submited to Journal of Fuzzy Mathematics).

Feb.1987.