SOME WEAK ORDER ON THE SET OF HOMOGENEOUS INTERVAL NUMBERS

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Let $\bar{\mathbb{R}} = [-\infty, +\infty]$. A weak order (see for example [1], [2]) on $\bar{\mathbb{R}}$ can be given in following manner.

Definition 1 [3] : Each mapping $\varrho : \bar{\mathbb{R}}^2 \to [0,1]$ such that

$$\varrho(x,y) \geqslant 1 - \varrho(y,x) \quad , \tag{1}$$

$$\varrho(x,y) + \varrho(z,x) \leqslant 1 \quad , \tag{2}$$

for each $(x,y,z) \in \bar{\mathbb{R}}^3$ with $y < z$, is called a fuzzy relation "less or equal" (FLE).

Definition 2 [3] : The mapping $\varrho_s : \bar{\mathbb{R}}^2 \to [0,1]$ given by the identity

$$\forall (x,y) \in \bar{\mathbb{R}}^2 \qquad \varrho_s(x,y) = 1 - \varrho(y,x) \tag{3}$$

where ϱ is a fixed FLE, is called a fuzzy relation "less than" (FLT) generated by FLE ϱ.

Additionaly we distinguish the following kinds of FLE.

Definition 3 [3] : Each FLE ϱ satisfying

$$\varrho(x,y) \wedge \varrho(y,x) < \frac{1}{2} \tag{4}$$

for each $(x,y) \in \bar{\mathbb{R}}^2$ such that $x \neq y$, is called quasi-anti-symetrical.

Definition 4 [3] : Each FLE ϱ satisfying the next conditions:

$$\forall \{x_n\} \downarrow x \qquad \{\varrho(\cdot,x_n)\} \downarrow \varrho(\cdot,x) \ , \tag{5}$$

$$\forall \{y_n\} \uparrow y \qquad \{\varrho(y_n,\cdot)\} \uparrow \varrho(y,\cdot) \tag{6}$$

is called a continuous from above FLE.

Definition 5 [3] : Any FLE ϱ unfuzzily bounds the real line if it satisfies

$$\forall x \in \bar{\mathbb{R}} \qquad \varrho(x,+\infty) = \varrho(-\infty,x) = 1 \tag{7}$$

Let us look on the set of interval numbers. By on interval we mean a closed bounded set of "real" numbers from $\bar{\mathbb{R}}$

$$[a,b] = \{x:\ a \leqslant x \leqslant b\} \ .$$

If A is an interval, we will denote its end points by $\underline{A}$ and $\bar{A}$. Thus $A = [\underline{A},\bar{A}]$. The family of all intervals will be indicated by $I(\bar{\mathbb{R}})$. We can extend the order relation, $\leqslant$, on $\bar{\mathbb{R}}$ to $I(\bar{\mathbb{R}})$ as follows:

$$A \leqslant B \quad \text{iff} \quad \underline{A} \leqslant \underline{B} \ \text{ and } \ \bar{A} \leqslant \bar{B} \ .$$

Moreover, we shall use the following "ordering" relation on $I(\bar{\mathbb{R}})$

$$A \prec B \quad \text{iff} \quad \bar{A} \leqslant \underline{B} \ .$$

A family of homogeneous interval numbers will be given by the mapping which is presented below.

Definition 6: Each mapping $\mathfrak{J}: \bar{\mathbb{R}} \rightarrow I(\bar{\mathbb{R}})$ having the next properties:

$$\mathfrak{J}(-\infty) = [-\infty,-\infty] \quad , \tag{8}$$
$$\mathfrak{J}(+\infty) = [+\infty,+\infty] \quad , \tag{9}$$
$$\forall x \in \mathbb{R} \qquad -\infty < \underline{\mathfrak{J}}(x) \leqslant \overline{\mathfrak{J}}(x) < +\infty \quad , \tag{10}$$
$$\forall x \in \mathbb{R} \qquad \overline{\mathfrak{J}}(x) - \underline{\mathfrak{J}}(x) = \Delta \quad , \tag{11}$$
$$\forall (x,y) \in \mathbb{R}^2 \qquad x < y \Rightarrow \mathfrak{J}(x) \leqslant \mathfrak{J}(y) \tag{12}$$

is called a homogeneous nondecreasing interval function.

Let $\mathfrak{J}$ be the fixed homogeneous nondecreasing interval function. It generates the homogeneous family of interval numbers $\mathfrak{J}(\overline{\mathbb{R}}) = \{A,\ \exists\ x \in \overline{\mathbb{R}},\ A = \mathfrak{J}(x)\}$. Take into account the measurable families of subsets in $\overline{\mathbb{R}}$, given by

$$\mathfrak{B}_{-\infty} = \{\emptyset,\ [-\infty,-\infty]\} \ ,$$
$$\mathfrak{B}_{+\infty} = \{\emptyset,\ [+\infty,+\infty]\} \ ,$$
$$\mathfrak{B}_x = \{A:\ \exists\ B \in \mathbb{B}^1\ \ A = \mathfrak{J}(x) \cap B\} \ ,$$

for each $x \in \mathbb{R}$ where $\mathbb{B}^n$ is the usual Borel field in $\mathbb{R}^n$. Since the measures with uniform distribution are well suited for the homogeneity of interval numbers, we propose to define such denumerable additive measures m_x on each $\mathfrak{B}_x$ $(x \in \overline{\mathbb{R}})$ that they fulfil:

$$\forall x \in \overline{\mathbb{R}} \qquad m_x(\mathfrak{J}(x)) = 1 \tag{13}$$
$$\forall (x,c) \in \mathbb{R} \times \mathfrak{J}(x) \qquad \Delta > 0 \Rightarrow m_x([\,\underline{\mathfrak{J}}(x), c\,[\,) = \frac{c - \underline{\mathfrak{J}}(x)}{\Delta} \tag{14}$$

Then, for each pair $(x,y) \in \mathbb{R}^2$, there exists the unique product measure m_{xy} on $\mathfrak{B}_{xy} = \{A,\ \exists\ B \in \mathbb{B}^2:\ A = (\mathfrak{J}(x) \times \mathfrak{J}(y)) \cap B\}$ [4] with

$$\forall (A,B) \in \mathfrak{J}(x) \times \mathfrak{J}(y) \qquad m_{xy}(A \times B) = m_x(A) \cdot m_y(B) \ . \tag{15}$$

Since $\mathfrak{B}_{-\infty}$ and $\mathfrak{B}_{+\infty}$ are two-elements families, the class of measurable subsets in $\mathfrak{J}(x) \times \mathfrak{J}(y)$ is given by $\mathfrak{B}_{xy} = \{A:\ \exists\ (B,C) \in \mathfrak{B}_x \times \mathfrak{B}_y:\ A = B \times C\}$ for each pair $(x,y) \in \mathbb{M}(\overline{\mathbb{R}}^2) = \overline{\mathbb{R}} \times \overline{\mathbb{R}} \setminus \mathbb{R} \times \mathbb{R}$. Thus the product measure m_{xy} is explicitly given by (15), for each $(x,y) \in \overline{\mathbb{R}}^2$.

In agreement with the intuition, we can say that $x \leqslant y$ in degree equal to the measure of the set $\{(t,s): t \in \mathfrak{I}(x),\ s \in \mathfrak{I}(y),\ t \leqslant s\}$ for each $(x,y) \in \overline{\mathbb{R}}^2$. Also, in like manner we can define a degree of $x < y$.

Definition 7: The mapping $\tilde{\varsigma} : \overline{\mathbb{R}}^2 \to [0,1]$ given by

$$\tilde{\varsigma}(x,y) = m_{xy}(\{(t,s): t \in \mathfrak{I}(x),\ s \in \mathfrak{I}(y),\ t \leqslant s\}) \tag{16}$$

for each $(x,y) \in \overline{\mathbb{R}}^2$, is called a order with interval imprecision (OII).

Definition 8: The mapping $\tilde{\varsigma}_s : \overline{\mathbb{R}}^2 \to [0,1]$ given by

$$\tilde{\varsigma}_s(x,y) = m_{xy}(\{(t,s): t \in \mathfrak{I}(x),\ s \in \mathfrak{I}(y),\ t < s\}) \tag{17}$$

for each $(x,y) \in \overline{\mathbb{R}}^2$, is called a strict order with interval imprecision (SOII).

It is very easy to check that if mapping $\mathfrak{I}: \overline{\mathbb{R}} \to I(\overline{\mathbb{R}})$ is degenerate (i.e. $\Delta = 0$) then the OII and SOII describe respectively the usual order relations $\leqslant$ and $<$ in $\overline{\mathbb{R}}$. Consider now the case when $\Delta > 0$. For any pair $(x,y) \in \overline{\mathbb{R}}^2$ with $x < +\infty$ and $y > -\infty$, using (13), (15), (16) and (17) we have:

$$\tilde{\varsigma}(+\infty,+\infty) = m_{+\infty+\infty}(\mathfrak{I}(+\infty) \times \mathfrak{I}(+\infty)) = 1 , \tag{18}$$

$$\tilde{\varsigma}(x,+\infty) = m_{x+\infty}(\mathfrak{I}(x) \times \mathfrak{I}(+\infty)) = 1 , \tag{19}$$

$$\tilde{\varsigma}(+\infty,x) = m_{+\infty x}(\emptyset \times \emptyset) = 0 , \tag{20}$$

$$\tilde{\varsigma}(y,-\infty) = m_{y-\infty}(\emptyset \times \emptyset) = 0 , \tag{21}$$

$$\tilde{\varsigma}(-\infty,y) = m_{-\infty y}(\mathfrak{I}(-\infty) \times \mathfrak{I}(y)) = 1 , \tag{22}$$

$$\tilde{\varsigma}(-\infty,-\infty) = m_{-\infty-\infty}(\mathfrak{I}(-\infty) \times \mathfrak{I}(-\infty)) = 1 , \tag{23}$$

$$\tilde{\varsigma}_s(+\infty,+\infty) = m_{+\infty+\infty}(\emptyset \times \emptyset) = 0 , \tag{24}$$

$$\tilde{\varsigma}_s(+\infty,x) = m_{+\infty x}(\emptyset\times\emptyset) = 0 \quad , \tag{25}$$

$$\tilde{\varsigma}_s(x,+\infty) = m_{x+\infty}(\mathcal{J}(x)\times\mathcal{J}(+\infty)) = 1 \quad , \tag{26}$$

$$\tilde{\varsigma}_s(-\infty,y) = m_{-\infty y}(\mathcal{J}(-\infty)\times\mathcal{J}(y)) = 1 \quad , \tag{27}$$

$$\tilde{\varsigma}_s(y,-\infty) = m_{y-\infty}(\emptyset\times\emptyset) = 0 \quad , \tag{28}$$

$$\tilde{\varsigma}_s(-\infty,-\infty) = m_{-\infty,-\infty}(\emptyset\times\emptyset) = 0 \quad . \tag{29}$$

From practical point-view, all above result are sensible. Nextly we have:

Theorem 1: The OII $\tilde{\varsigma}$ and SOII $\tilde{\varsigma}_s$ fulfil the condition (3).

Proof: For any pair $(x,y)\in \mathbb{M}(\bar{\mathbb{R}})$, the condition (3) follows from the identities (18) - (29). If $(x,y)\in\mathbb{R}^2$ then the Fubini's Theorem [4] says $m_{xy} = m_{yx}$. Hence

$$\begin{aligned}\tilde{\varsigma}_s(x,y) &= m_{xy}(\{(t,s):\ t\in\mathcal{J}(x),\ s\in\mathcal{J}(y),\ t<s\}) = \\ &= m_{xy}((\mathcal{J}(x)\times\mathcal{J}(y))\setminus\{(t,s):\ t\in\mathcal{J}(x),\ s\in\mathcal{J}(y),\ t\geqslant s\}) = \\ &= 1 - m_{xy}(\{(t,s):\ t\in\mathcal{J}(x),\ s\in\mathcal{J}(y),\ s\leqslant t\}) = \\ &= 1 - m_{yx}(\{(s,t):\ s\in\mathcal{J}(y),\ t\in\mathcal{J}(x),\ s\leqslant t\}) = \tilde{\varsigma}(y,x)\ . \blacksquare\end{aligned}$$

Theorem 2: If $(x,y)\in\mathbb{R}^2$ and $\mathcal{J}(x)\cap\mathcal{J}(y)\neq\emptyset$ then

$$\tilde{\varsigma}(x,y) = \begin{cases} 1 - \dfrac{1}{2\Delta^2}(\bar{\mathcal{J}}(x) - \underline{\mathcal{J}}(y))^2 & x\leqslant y \\ \dfrac{1}{2\Delta^2}(\bar{\mathcal{J}}(y) - \underline{\mathcal{J}}(x))^2 & x>y \end{cases} \tag{30}$$

$$\tilde{\varsigma}_s(x,y) = \begin{cases} 1 - \dfrac{1}{2\Delta^2}(\bar{\mathcal{J}}(x) - \underline{\mathcal{J}}(y))^2 & x<y \\ \dfrac{1}{2\Delta^2}(\bar{\mathcal{J}}(y) - \underline{\mathcal{J}}(x))^2 & x\geqslant y \end{cases} \tag{31}$$

Proof: The identity (30) follows from (14) and (16). The identity (31) is an immediate consequence of (3) and (30). ■

Theorem 3: If $(x,y)\in \mathbb{R}^2$ and $\mathfrak{I}(x)\prec\mathfrak{I}(y)$ then $\tilde{\varsigma}(x,y)=1$.

Proof: $\varsigma(x,y)=m_{xy}(\{(t,s): t\in\mathfrak{I}(x),\ s\in\mathfrak{I}(y),\ t\leqslant s\})=$
$=m_{xy}(\mathfrak{I}(x)\times\mathfrak{I}(y))=1$ ■

Theorem 4: If $(x,y)\in \mathbb{R}^2$ and $\mathfrak{I}(y)\prec\mathfrak{I}(x)$ then $\tilde{\varsigma}(x,y)=0$.

Proof: $\tilde{\varsigma}(x,y)=m_{xy}(\{(t,s): t\in\mathfrak{I}(x),\ s\in\mathfrak{I}(y),\ t\leqslant s\})\leqslant$
$m_{xy}([\bar{\mathfrak{I}}(x),\bar{\mathfrak{I}}(x)]\times\mathfrak{I}(y))=0$. ■

Theorem 5: Any OII $\tilde{\varsigma}$ is a FLE.

Proof: Using the Theorem 2 we get

$$1-\tilde{\varsigma}(x,y)=\tilde{\varsigma}_s(y,x)=m_{yx}(\{(t,s): t\in\mathfrak{I}(y),\ s\in\mathfrak{I}(x),\ t<s\})\leqslant$$
$$\leqslant m_{yx}(\{(t,s): t\in\mathfrak{I}(y),\ s\in\mathfrak{I}(x),\ t\leqslant s\})=$$
$$=\tilde{\varsigma}(y,x)$$

for each pair $(x,y)\in\mathbb{R}^2$. So, the condition (1) holds.

If $(x,y)\in \mathbb{M}(\mathbb{R}^2)$ then the condition (2) follows from the identities (18) - (29). Suppose now that $(x,y)\in\mathbb{R}^2$ and $z=$ $=+\infty$. Then, using (20), we obtain

$$\tilde{\varsigma}(x,y)+\tilde{\varsigma}(+\infty,x)=\tilde{\varsigma}(x,y)\leqslant 1\ .$$

For the case, when $(x,y,z)\in\mathbb{R}^3$ and $y<z$, we get:

- if $\mathfrak{I}(x)\prec\mathfrak{I}(z)$ then, using the Theorem 4,
$$1-\tilde{\varsigma}(z,x)=1\geqslant\tilde{\varsigma}(x,y)\quad;$$
- if $\mathfrak{I}(y)\prec\mathfrak{I}(x)$ then, in like manner as above,
$$\tilde{\varsigma}(x,y)=0\leqslant\tilde{\varsigma}_s(x,z)=1-\tilde{\varsigma}(z,x)\quad;$$
- otherwise, we have $\bar{\mathfrak{I}}(x)>\underline{\mathfrak{I}}(z)$ and $\bar{\mathfrak{I}}(y)>\underline{\mathfrak{I}}(x)$.

Thus for each $(x,y,z)\in\mathbb{R}^3$ with $x\leqslant y<z$ we obtain $\bar{\mathfrak{I}}(z)\geqslant$ $\geqslant\bar{\mathfrak{I}}(y)\geqslant\bar{\mathfrak{I}}(x)>\underline{\mathfrak{I}}(z)\geqslant\underline{\mathfrak{I}}(y)\geqslant\underline{\mathfrak{I}}(x)$. So, $\mathfrak{I}(x)\cap\mathfrak{I}(y)\neq\emptyset$ and

$J(x) \cap J(z) \neq \emptyset$. Hence, by the identity (30), we get:

$$\tilde{\varsigma}(x,y) + \tilde{\varsigma}(z,x) = 1 - \frac{1}{2\Delta^2}(\bar{J}(x) - \underline{J}(y))^2 + \frac{1}{2\Delta^2}(\bar{J}(x) - \underline{J}(z))^2 =$$

$$= 1 + \frac{1}{2\Delta^2}\left((\bar{J}(x) - \underline{J}(z))^2 - (\bar{J}(x) - \underline{J}(y))^2\right) \leqslant 1 \ .$$

Moreover if $y < x \leqslant z$ then $\bar{J}(x) > \underline{J}(z) \geqslant \underline{J}(x) \geqslant \underline{J}(y)$ and $\bar{J}(z) \geqslant \bar{J}(x) \geqslant \bar{J}(y) > \underline{J}(x) \geqslant \underline{J}(y)$. It shows that $J(x) \cap J(y) \neq \emptyset$ and $J(x) \cap J(z) \neq \emptyset$. Thus, by (30), we obtain:

$$\tilde{\varsigma}(x,y) + \tilde{\varsigma}(z,x) = \frac{1}{2\Delta^2}(\bar{J}(y) - \underline{J}(x))^2 + \frac{1}{2\Delta^2}(\bar{J}(x) - \underline{J}(z))^2 \leqslant$$

$$\leqslant \frac{1}{2\Delta^2}(\bar{J}(x) - \underline{J}(x))^2 + \frac{1}{2\Delta^2}(\bar{J}(z) - \underline{J}(z))^2 = 1$$

If $y < z < x$ then $\bar{J}(x) > \underline{J}(z) \geqslant \underline{J}(y)$ and $\bar{J}(x) \geqslant \bar{J}(z) \geqslant$ $\geqslant \bar{J}(y) > \underline{J}(x) \geqslant \underline{J}(z) \geqslant \underline{J}(y)$. Therefore, $J(x) \cap J(y) = \emptyset$ and $J(x) \cap J(z) \neq \emptyset$. Using (30), we get:

$$\tilde{\varsigma}(x,y) + \tilde{\varsigma}(z,x) = \frac{1}{2\Delta^2}(\bar{J}(y) - \underline{J}(x))^2 + 1 + \frac{1}{2\Delta^2}(\bar{J}(z) - \underline{J}(x))^2 =$$

$$= 1 + \frac{1}{2\Delta^2}\left((\bar{J}(y) - \underline{J}(x))^2 - (\bar{J}(z) - \underline{J}(x))^2\right) \leqslant 1 \ .$$

The proof is ended. ∎

The last theorem together with the Theorem 1 shows that any SOII is a FIT generated by OII. Furthermore, we have:

Theorem 6: Any OII unfuzzily bounds the real line.

Proof: See (18), (19), (22) and (23). ∎

Theorem 7: The OII $\tilde{\varsigma}$ is quasi-antisymmetrical in $J(\mathbb{R})$ iff the interval mapping $J: \mathbb{R} \to I(\mathbb{R})$ has the property:

$$\forall (x,y) \in \mathbb{R}^2 \qquad x \neq y \Rightarrow J(x) \neq J(y) \ . \qquad (32)$$

Proof: If $x \neq y$ then $x<y$ or $x>y$. Suppose now that $x>y$. Then, for the case $\mathcal{J}(y) \prec \mathcal{J}(x)$, the Theorem 4 shows $\tilde{\varrho}(x,y) = 0$. Otherwise we have $\underline{\mathcal{J}}(y) \leqslant \underline{\mathcal{J}}(x) < \overline{\mathcal{J}}(y) \leqslant \overline{\mathcal{J}}(x)$ and $\underline{\mathcal{J}}(y) < \underline{\mathcal{J}}(x)$ or $\overline{\mathcal{J}}(y) < \overline{\mathcal{J}}(x)$. Thus, by (30), we get:

$$\tilde{\varrho}(x,y) = \frac{1}{2\Delta^2}\left(\overline{\mathcal{J}}(y) - \underline{\mathcal{J}}(x)\right)^2 < \frac{1}{2\Delta^2}\left(\overline{\mathcal{J}}(x) - \underline{\mathcal{J}}(x)\right)^2 = \frac{1}{2}$$

or

$$\tilde{\varrho}(x,y) = \frac{1}{2\Delta^2}\left(\overline{\mathcal{J}}(y) - \underline{\mathcal{J}}(x)\right)^2 < \frac{1}{2\Delta^2}\left(\overline{\mathcal{J}}(y) - \underline{\mathcal{J}}(y)\right)^2 = \frac{1}{2}$$

So, (32) is sufficient for the quasi-antisymmetry. Suppose now that there exists such pair $(x,y) \in \overline{\mathbb{R}}^2$ that $x>y$ and $\mathcal{J}(x) = \mathcal{J}(y)$. Thus

$$\tilde{\varrho}(x,y) = \frac{1}{2\Delta^2}\left(\overline{\mathcal{J}}(y) - \underline{\mathcal{J}}(x)\right)^2 = \frac{1}{2\Delta^2}\Delta^2 = \frac{1}{2}$$

We have seen that (32) is necessary for the quasi-antisymmetry. ∎

Theorem 7: If the interval function $\mathcal{J}: \overline{\mathbb{R}} \to I(\overline{\mathbb{R}})$ is continuous (i.e. $\underline{\mathcal{J}}: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ and $\overline{\mathcal{J}}: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are continuous) then OII ϱ is continuous from above.

Proof: Let $\{x_n\}$ be fixed sequence tending from above to $x \in \overline{\mathbb{R}}$. If $x = +\infty$ then $x_n = +\infty$ for each $n \in \mathbb{N}$. By (18) and (19) we get (5) because $\tilde{\varrho}(\cdot, x) = \tilde{\varrho}(\cdot, x_n) = 1$. Assume now that $x < +\infty$. Of course there exists such N_1 that $x_n < +\infty$ for each $n \geqslant N_1$.

Using (22) we get $1 = \tilde{\varrho}(-\infty, x_n) = \tilde{\varrho}(-\infty, x) = 1$. So,

$$\left\{\tilde{\varrho}(-\infty, x_n)\right\} \downarrow \tilde{\varrho}(-\infty, x) \quad . \qquad (*)$$

If $y \in \mathbb{R}$ and $\mathcal{J}(y) \prec \mathcal{J}(x)$ then $\mathcal{J}(y) \prec \mathcal{J}(x_n)$ for each positive integer n . Taking into account the Theorem 3 we obtain

$$\{\tilde{\varrho}(y,x_n)\} \downarrow \tilde{\varrho}(y,x) \qquad (**)$$

because $\tilde{\varrho}(y,x_n) = \tilde{\varrho}(y,x) = 1$.

If $y \in \mathbb{R}$ and $\mathfrak{J}(y) \cap \mathfrak{J}(x) \neq \emptyset$ then there exists such positive integer $N_2 \geqslant N_1$ that $\mathfrak{J}(y) \cap \mathfrak{J}(x_n) \neq \emptyset$ for each $n \geqslant N_2$. Using (30) we obtain $(**)$ from continuity of $\mathfrak{J}$.

If $y \in \mathbb{R}$ $\mathfrak{J}(x) \prec \mathfrak{J}(y)$ and $\overline{\mathfrak{J}}(x) < \underline{\mathfrak{J}}(y)$ then there exists such $N_3 \geqslant N_1$ that $\overline{\mathfrak{J}}(x_n) \leqslant \underline{\mathfrak{J}}(y)$. Then the condition $(**)$ follows from the Theorem 4 because $\tilde{\varrho}(y,x) = \tilde{\varrho}(y,x_n) = 0$.

Using (20) we get $0 = \tilde{\varrho}(+\infty,x_n) = \tilde{\varrho}(+\infty,x) = 0$. So,

$$\{\tilde{\varrho}(+\infty,x_n)\} \downarrow \tilde{\varrho}(+\infty,x) .$$

The last result along with $(*)$ and $(**)$ puts an end to the proof of (5). In like manner we can checlo the condition (6). ■

All foregoing considerations can be generalized for the case when the condition (14) is replaced by the following one

$$\forall (x,c) \in \mathbb{R} \times \mathfrak{J}(x) \qquad \Delta > 0 \quad m_x([\underline{\mathfrak{J}}(x), c[) = F\left(\frac{c - \underline{\mathfrak{J}}(x)}{\Delta}\right)$$

where $F: [0,1] \to [0,1]$ is a continuous nondecreasing function fulfilling $F(0) = 0$ and $F(1) = 1$.

On the other side, the above considerations cannot be generalized for the case when the interval function $\mathfrak{J}: \overline{\mathbb{R}} \to I(\overline{\mathbb{R}})$ is not homogeneous (i.e. there exists such pair $(x,y) \in \mathbb{R}^2$ that $\overline{\mathfrak{J}}(x) - \underline{\mathfrak{J}}(x) \neq \overline{\mathfrak{J}}(y) - \underline{\mathfrak{J}}(y)$) .

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