THE HYPOTHESIS ON THE NUMBER OF LOWER SOLUTIONS OF A FUZZY RELATION EQUATION

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In this paper, we have proved it is correct that the hypothesis which the number of lower solutions of a fuzzy relation equation on a finite set is not more than 2^n-1, which was proposed by E.Czogala, J.Drewniak, W.Pedeycz in [1]. Something more, we introduce a precise estimation on the number of lower solutions.

If $A \in F(X)=\{A \mid A : X \to [0,1]\}$ is unknown, then the following equation

$$A\circ R(y_j)=\bigvee_{i=1}^{n}\left[r_{ij}\wedge A(x_i)\right]=B(y_j),\ j\in N=\{1,2,\dots,n\} \quad (1)$$

is called a max- min composition fuzzy relation equation. We write $(b_1,b_2,\dots,b_n)$ instead of the $B(y_j)$. We assume $b_1\leq b_2\leq\dots\leq b_n$ and we write Δ_i to mear $\wedge\{b_j \mid r_{ij}>b_j,\ j\in N\}$, $i\in N$. Specialiy, $\wedge\phi=1$. We now define $\bar{R}=(\bar{r}ij)_{n\times n}$ as follows: $\bar{r}_{ij}=b_j$ when $b_j\leq r_{ij}$ and $b_j\leq\Delta_i$, otherwise, $\bar{r}_{ij}=0$. The transpose of $\bar{R}$ is denoted by $A=(a_{ij})_{n\times n}$, namely $a_{ij}=\bar{r}_{ji}$. This set $\{j(i) \mid j(i)\in N,\ a_{i\,j(i)}\neq 0\}$ is denoted by F_i and we denote $\bigcup_{i=1} F_i$ by the notation F. For each $l\in N$ we use the notation $Q(l)$ to denote the set $\{k \mid k\in N,\ a_{k\,j(k)}=a_{l\,j(l)},\ \forall j(l)\in F_l,\ \forall j(k)\in F_k\}$. We write l_1 and l_2 to denote $\min_k Q(l)$ and $\max_k Q(l)$ respectively. If $l_2-l_1\geq 2$, then we give

Definition 1. If th-ere are u and v to satisfty the following conditions for $u,\ v\in N$:

(a) $a_{l_1u}=a_{l_1v}$, $a_{l_2u}=a_{l_2v}$, for $u, v \in F_{l_1} \cap F_{l_2}$,

(b) if $u \in F_t$, then $v \in F_t$, for each $t \in N$ and $l_1 < t < l_2$,

(c) there is $t_0 \in N$, $l_1 < t_0 < l_2$ such that $v \in F_{t_0}$ but $u \notin F_{t_0}$,

then we said that a_{t_0v} is a redundant element of the element a_{l_2u} with respect to vth column in A.

Definition 2. Let $a \neq 0$. Then this submatrix $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ of the matrix A is called an alternate submatrix of A.

We write $M \times F$ to denote the set $\bigcup_{i=1}^{n} \{i\} \times F_i$. Let $* : M \times F \longrightarrow N$ be a map. I shall use the notation $i*j$ to denote $*((i,j))$ for each $(i,j) \in M \times F$. If this operation $i*j$ already have been defined, then $i*j$ is said to be an admissible element.

Definition 3. Let there be the representatives $t_{i+1}=t_i*j(t_i)$, $i=1,2,\ldots,k$, $t_1=n$ and the $t_i*j(t_i)$ is an admissible element. Then the sequence $(j(t_1), j(t_2),\ldots, j(t_k))$ is said to be a path at the initial point of $(n,j(n))$ and at the finishing point of $(t_k,j(t_k))$. This $j(t_i)$ is said to be ith nodal point. For each $(i,j) \in N \times F$ we define

$$i*j=\bigvee \{ x \mid x \in (A_{ij}-E_{ij}) \}, \text{ specially, } \bigvee \phi =0,$$

in which the A_{ij} and E_{ij} are defined by the equations respectively

$$A_{ij}=\{ p \mid p \in N,\ p<i,\ a_{pj}=0 \} \text{ and } E_{ij}=\{ p \mid p \in A_{ij}, F_p \cap G(i)=\phi \},$$

where $G(i)=\{ m \mid m=j(t_s),\ s=1,2,\ldots,k, t_1=n; j(t_1),j(t_2),\ldots,j(t_k)$ are nodal points of the path at the initial point of $(n,j(n))$ and at the finishing point of $(t_k,j(t_k))$ and $i=t_k*j(t_k)\}$. Obviously, $G(n)=\phi$.

Definition 4. Let the value of $i*j$ is defined by the path p. The $\bar{p}$ is a path at the finishing point of the (i,j), which nodal points are one with p all but one (i,j). Then we said that $\bar{p}$ is a nature path with respect to the path p. We also define $i*j*k=(i*j)*k$. We write $M*F$ to denote the set $\{ i*j \mid \text{for } \forall (i,j) \in M \times F \}$.

Let $q: N*F \rightarrow N \cup \{0\}$ be a map satisfying the following conditions

(e) $q(0)=1$, (f) $q(i*j*k)=0$ when $a_{i*j\,k}$ is a redundant element with respect to a_{ij}; (g) $q(i*j)=\sum_{k \in F_{i*j}} q(i*j*k)$, in which $i*j*k$ is defined by the path that is a nature path with respect to the path defined the $i*j$. We write D_0 to mean the number of the alternate submatrixs of A. Then we have

Theorem 1. Let w be the set of all lower solution of relation equation (1). Then we have

$$\text{(h)} \quad q(n)-D_0 \leq \text{card } w \leq q(n),$$

$$\text{(i)} \quad \text{card } w \leq 2^n-1 \text{ , where } q(n)=\sum_{j(n) \in F_n} q(n*j(n)).$$

Corollary. Let $b_1 < b_2 < \dots < b_n$. Then card $w=q(n)$.

Proof. Obviously, $l_2-l_1 < 2$. Hence $D_0=0$.

2. Proof of the theorem 1.

In order to prove theorem 1 we first introduce a new method concerning obtaining the all lower solutions of the equation (1). In fact, acco_rding to algorithm TA that we will introduce the following we can obtain a lower solution of the equation (1) by taking the elements from nth row in the matrix A untie the 1th row, which are different in the columns.

(A) Algorithm TA.

Step 1. Taking an element a_{ij} of the matrix A and one is treated as jth component of some lower solution.

Step 2. If $a_{ij}=0$, then go back to step 1. If there exists If $a_{ij} \neq 0$ and $a_{qj}=0$ for each $q \in \{1,2,\dots,i-1\}$, then go back to step 1. If there exists $q \in \{1,2,\dots,i-1\}$ such that $a_{qj} \neq 0$ then we first have to make $a_{qj}=0$ for each $j \in N$ and then go back to step 1.

(B) . BUle of algorithm TA.

(i) If there exists $q_0 \in \{1,2,\ldots,i-1\}$ such that $a_{q_0 j}=0$ for the step 2, then we only cam take the nor-zero element beloning to q_0th row after go back to step 1.

(ii) If we have taked the element a_{ij} and have ~~treq~~ treated as a component of some lower solution, then we cannot take the redundant of the element of the element a_{ij} as component of this lower solution.

It is noteworthy that we hardly more than need to take no-zero element for step 1.

If we take the elements in every way possible under the algorithm TA, then we can obtain the all lower solutions of the equation (1). Further, we can prove the following lemmas:

Let $\alpha=(a_1,a_2,\ldots,a_n)$ be a sub-set of the F(X). Write

$\beta(\alpha)=\{ a_i \mid a_i\neq 0,\ a_i$ is a component of the $\alpha\}$:

$T=\{ b_j \mid b_j=B(y_j),\ j\in N \}$:

$\delta_j=\{b \mid b\in T,$ there exists $i\in N$ such that $b\wedge r_{ij}=b_j\}$.

Lemma 1. If there exists j such that $a_i\in\delta_j$ for each $a_i\in\beta(\alpha)$ and $\beta(\alpha)\cap\delta_K\neq\phi$ for each $k\in N$, then α is a solution of the equation (1).

This proof to ~~le~~lemma 1 is easily.

We write

$\xi(\alpha)=\{j \mid \text{card } \beta(\alpha)\cap\delta_j=1,\ j\in N,\ \alpha$ is a solution of the (1)$\}$; $W(\alpha_0)=\{\alpha \mid \alpha$ is a solution of the (1) and $\xi(\alpha)\subset\xi(\alpha_0)$ or $\xi(\alpha_0)\subset\xi(\alpha)\}$ where α_0 is a solution of the (1).

Lemma 2. Let α_0 be a solution of the equation (1). Then α_0 is a lower solution if and only if card $\xi(\alpha_0)\geq$ card $\xi(\alpha)$ for each $\alpha\in W(\alpha_0)$.

Proof. It is clear from definition of the lower solution and lemma 1.

Lemma 3. Let a subset η of $F(X)$ be obtained through using the algorithm TA. Then η is a lower solution.

Proof. By step 2 of the algorithm TA we get $\eta \wedge \delta_j \neq \phi$, $j \in N$, and the condition for lemma2 can be satisfied. Hence, η is a lower solution by lemma2.

Lemma 4. Through using the algorithm TA. We can obtain the all lower solutions of the equation (1).

Proof. Let us assume that there exists a lower solution $A^*=(a_1,a_2,\ldots,a_n)$ which we cannot obtain by using the algorithm TA. Without loss of generality, let $a_i \neq 0$, $i \in N$ and $a_i \wedge r_{ij}=b_j$ and $r_{ij} \geqslant b_j$ must hold. otherwise, A^* is not a lower solution. But for this case either $a_{ij}=0$ or $a_{ij}=b_j$ must hold. If $a_{ij}=0$, then $r_{ij} > b_j > \Delta_i$. Let $\Delta_i = b_k$ then $r_{ik} > b_k$, it follows that $b_j \wedge r_{ik} > b_k \wedge r_{ik} = b_k$. Hence, A^* is not a solution of the equation (1). Hence, only, $a_{ij}=b_j$. If we can make $a_{ij}=0$ by using the algorithm TA, then A^* is not a lower solution. Thus we get the contradiction because we let A^* be a lower solution ago.

Now let us return to prove theorem 1. We first consider the case (h) on theorem 1.

If $b_1 < b_2 < \cdots < b_n$, then D_0 is equal to zero and there are no redundant element for each a_{ij}. We write $n*j_0(n)=t_i*j_0(t_i)=t_{i+1}$, $i=1,2,\ldots,k$, in which $j_0(t_i) \in F_{t_i}$ and $a_{t_i j_0(t_i)} \neq 0$. Thus,

$$
\begin{aligned}
q(n) &= \sum_{k \in F_n} q(n*k) \\
&= \cdots + q(n*j_0(n)) + \cdots \\
&= \cdots + q(t_1*j_0(t_1)) + \cdots \\
&= \cdots \\
&= \cdots + q(t_i*j_0(t_i)) + \cdots \\
&= \cdots + 1 + \cdots .
\end{aligned}
$$

Hence, according to algorithm TA here there is a rule f which assigus to each finite sequence ($q(n*j_0(n))$, $q(t_1*j_0(t_1))$,...,$q(t_i*j_0(t_i))$, ...) a unique lower solution (..., $a_{nj_0(n)}$, $a_{t_1j_0(n)}$, ~~a~~..., $a_{t_ij_0(t_i)}$), ...) .

Conversoly, it is clear that this correspondence is a surjection by the algorithm TA. Hence, we have

$$q(n)= \text{card } w.$$

In addition, when $b_1 \leqslant b_2 \leqslant \dots \leqslant b_n$, it is possible that $D_0 \neq 0$. But a necessary condition that we can obtain the very same lower solutions by using algorithm TA is that the no-zero elements of some alternate submatrix of the matrix A had been taked as a different component blonging to some lower solution. Hence, so as we have

$$q(n)-D_0 \lesssim \text{card } w \lesssim q(n).$$

Now to prove the case (1) for the theorem 1. We write

$$\beta(A)= \{ a_{ij} \mid a_{ij} \neq 0,\ a_{ij} \text{ is an element of the matrix } A \};$$

$$\beta(\alpha)= \{ a_{ij} \mid a_{ij} \neq 0,\ a_{ij} \text{ is jth component of the } \alpha \}$$

where α is a lower solution.

Lemma 5. If card $\beta(A) \lesssim 2n$, then card $w \lesssim 2^n-1$. If card $\beta(A) > 2n$, then $\beta(\alpha) < n$.

Proof. If card $\beta(A) \lesssim 2n$, then card $w \lesssim \overbrace{2 \times 2 \times \dots \times 2}^{n}$. Furthermore, by the definition of a lower solution, if $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in W$, then at least there exists a $\alpha' = (\alpha_1', \alpha_2', \cdots \alpha_i', \cdots \alpha_n') \notin W$, where $\alpha_i' > \alpha_i$, $\alpha_i \in \beta(A)$, hence, we have card $w \lesssim 2^n-1$.

Now assume card $\beta(A) > 2n$. When we use the algorithm TA to find a lower solution, according to the case (i) for the rule of the algorithm TA we can know that the number of components which are equal to zero for the lower solution and this times when we use this step 2 for the algorithm TA and this case $a_{qj} \neq 0$ have appeared are equal to one another. But a necessary and sufficient condition to the case $a_{qj} \neq 0$ cannot appear is that, if only $(i\ j) \neq (k\ l)$ then $a_{kl}=0$ for each $(k\ l) \in N \times N$, $(i\ j) \in N \times N$.

WE assume that there exists k columns which satisfies this condition as above for the matrix A. We have

$$n^2-(kn+k(n-K)-k) \geq \text{card } \beta(A) > 2n, \ k \leq n.$$

Solving this inequality we can obtain

$$K \leq \left[n-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{4n+1}\right],$$

wh~~e~~ere denoted by $[x]$ the greatest integer not greater than x . It follows that at least there exists $n-[n-\frac{1}{2}-\frac{1}{2}\sqrt{4n+1}$ columns in the matrix A such that if only we take the elements belonging to there columns, then it will appear for this case $a_{qj} \neq 0$ when we use the step 2 of the algorithm TA. Hence, we have

$$\text{Card } \beta(\alpha) \leq \left[n-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{4n+1}\right] + \left[\tfrac{1}{2}(n-(n-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{4n+1}))\right]$$
$$< \left[n-\tfrac{1}{4}-\tfrac{1}{4}\sqrt{4n+1}\right] < n .$$

by the lemma 5 it is possible for us to define the vectors the following for each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in W$:

$$\alpha' = (\alpha_1', \alpha_2', \ldots, \alpha_n')$$
$$\alpha'' = (\alpha_1'', \alpha_2'', \ldots, \alpha_n'')$$

such that:

$$\alpha_j' = \begin{cases} 1 & \text{if } \alpha_j = a_{ij},\ a_{ij} \in \beta(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_i'' = \begin{cases} 1 & \text{if } \alpha_j = a_{ij},\ a_{ij} \in \beta(\alpha),\ j \in N, \\ 0 & \text{Otherwise} \end{cases} .$$

We also define the vectors $T_1(\alpha)$ and $T_2(\alpha)$ by equations $T_1(\alpha) = (\alpha', \alpha'')$ and $T_2 = (\alpha'', \alpha')$ for each $\alpha \in w$ respectively. Let $v = \{1, 0\}$. then $T_1(\alpha), T_2(\alpha) \in v^n \times v^n$. According to algorithm TA we can know that if $\alpha \neq \beta$ then $T_1(\alpha) \neq T_1(\beta)$ and $T_2(\alpha) \neq T_2(\beta)$ for all $\alpha, \beta \in w$.

Some sets that we will use the following for which we reserve special notation are:

$$D(\alpha') = \{\gamma \mid \gamma = (\alpha', \gamma_{n+1}, \cdots, \gamma_{2n}) \in V^n \times V^n\},$$

$$D(\alpha'') = \{s \mid s = (\alpha'', s_{n+1}, \cdots, s_{2n}) \in V^n \times V^n\},$$

$$D(\alpha_0') = \{t \mid t = (\underbrace{0, 0, \cdots, 0}_{n}, t_{n+1}, \cdots, t_{2n}) \in V^n \times V^n\},$$

Obviously, the $T_1(\alpha) \in D(\alpha')$ and the $T_2(\alpha) \in D(\alpha'')$. By the binary notation we have

$$\text{card } D(\alpha') = \text{card } D(\alpha'') = \text{card } D(\alpha_0') = 2^n .$$

Hence, we can determine a unique subset $D(\alpha')$ or $D(\alpha'')$ of the $v^n \times v^n$ for each $\alpha \in w$. Further, if $\alpha \neq \beta$, then either $\alpha' \neq \beta'$ or $\alpha'' \neq \beta''$. Thus, it is clear that eith~~r~~er $D(\alpha') \cap D(\beta') = \phi$ or $D(\alpha'') \cap D(\beta'') = \phi$. ~~T~~ Again, since $\alpha = (0,0,\ldots,0) \notin w$. Hence, we have

$$\text{card } w \leq \frac{1}{2^n} (\text{ card } v^n \times v^n - \text{card } D(\alpha_0'))$$

$$= \frac{1}{2^n} (2^{2n} - 2^n)$$

$$= 2^n - 1 .$$

3. Example.

Example 1. Iet us consider a fuzzy relation equation as followe:

$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \circ R = (0.3, 0.4, 0.5, 0.6, 0.7, 0.8).$

$$\text{where } (\bar{R})^t = A = \begin{bmatrix} 0 & 0.3 & 0 & 0 & 0.3 & 0 & 0.3 \\ 0.4 & 0 & 0 & 0.4 & 0 & 0.4 & 0.4 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.6 & 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0 & 0 & 0 & 0.7 & 0.7 \\ 0.8 & 0 & 0.8 & 0 & 0.8 & 0 & 0.8 \end{bmatrix}$$

Obviously, $F_1=\{2,5,7\}$, $F_2=\{1,4,6,7\}$, $F_3=\{2,4,\cancel{6}\ 7\}$, $F_4=\{1,4\}$, $F_5=\{1,2,6,7\}$, $F_6=\{1,3,5,7\}$.

(1), We take $n=6$ and $j(6)=1$. Since $6*1=3$, hence $\{3\}*F_3=\{3*2,\ 3*4,\ 3*7\}$; for $a_{32}\neq 0$ we have $A_{32}=\{2\}$, $G(3)=\{1\}$; $E_{32}=\{2\}$, hence $3*2=0$. In the same way as above we get $3*4=1$; $\{1\}*F_1=\{1*2,\ 1*5,\ 1*7\}=\{0\}$; $3*7=0$. Hence, finally, we have

$$\begin{aligned} q(6*1)&=q(3*2)+q(3*4)+q(3*7)\\ &=q(0)+q(1*2)+q(1*5)+q(1*7)+q(0)\\ &=5q(0)=5\,. \end{aligned}$$

(2), Take $n=6$ and $j(6)=3$. Using as above way we get the follows:

$$6*3=5,\ \{5\}*F_5=\begin{cases} 5*1=3,\ \{3\}*F_3= \begin{cases} 3*2=0\\ 3*4=1 \Rightarrow \begin{cases}1*2=0\\ 1*5=0\\ 1*7=0\end{cases}\\ 3*7=0 \end{cases}\\ 5*2=4,\ \{4\}*F_4= \begin{cases} 4*1=0\\ 4*4=0 \end{cases}\\ 5*6=4,\ \{4\}*F_4= \begin{cases} 4*1=3, \Rightarrow \begin{cases} 3*2=0\\ 3*4=1 \Rightarrow \begin{cases}1*2=0\\ 1*5=0\\ 1*7=0\end{cases}\\ 3*7=0 \end{cases}\\ 4*4=1 \Rightarrow \begin{cases}1*2=0\\ 1*5=0\\ 1*7=0\end{cases} \end{cases}\\ 5*7=4,\ \{4\}*F_4= \begin{cases} 4*1=0\\ 4*4=0 \end{cases} \end{cases}$$

Hence, $q(6*3)=\sum q(0)=17$. Similarly, we can obtain that $q(6*5)=10$ and $q(6*7)=2$. Hence we have

$$q(n)=q(6*1)+q(6*3)+q(6*5)+q(6*7)$$
$$=5+17+10+2$$
$$=34 .$$

References

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