FUZZY EQUATION SOLVING

Wang Aimin   Wang Qingyin   Yu Wanzhenh   Zhao Hanqing
Hebei Coal Mining and Civil engineering College.Handan City. Hebei Prov.,China

(Abstract).This article covers three parts;(1)the solution of the special fuzzy equation;(2)the solution of the general fuzzy equation;(3)the solution of equation $(R)\circ(X)=(S)$ The aim is to seek the generally simple solution of fuzzy equation

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The solution of the comprehensive judgements of the converse problems often consist in the solution of the fuzzy equation as follows.

$$(X)_{m\times n}\circ(R)_{n\times k}=(S)_{m\times k} \tag{1}$$

The given solution of this problem has been more complecated,so in this article we shall give a few new solving methods to the different forms of the equationin so that we can find a simple method of solving this fuzzy equation, making a trial for this purpose.

I. The solution of the special fuzzy equation

In equation (1),if

$$(R)_{n\times k}=(S)_{m\times k}$$

then (1) is called special fuzzy equation,suppose

$$R=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}$$

in it $r_{ij}\in[0,1]$,at this time the equation is

$$\begin{bmatrix} x_{11} & x_{12} & \cdots x_{1m} \\ x_{21} & x_{22} & \cdots x_{2m} \\ \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots x_{mm} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots r_{1n} \\ r_{21} & r_{22} & \cdots r_{2n} \\ \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots r_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & \cdots r_{1n} \\ r_{21} & r_{22} & \cdots r_{2m} \\ \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots r_{mn} \end{bmatrix} \qquad (2)$$

To equation (2) we have following theorems

Theorem 1.

1°,

$$(X) = \begin{bmatrix} [\bigvee_{j=1}^{n} r_{1j}, 1] & [0, \bigwedge_{j=1}^{n} r_{1j}] \cdots & [0, \bigwedge_{j=1}^{n} r_{1j}] \\ [0, \bigwedge_{j=1}^{n} r_{2j}] & [\bigvee_{j=1}^{n} r_{2j}, 1] \cdots & [0, \bigwedge_{j=1}^{n} r_{2j}] \\ \cdots\cdots & \cdots\cdots \quad \cdots & \cdots \\ [0, \bigwedge_{j=1}^{n} r_{mj}] & [0, \bigwedge_{j=1}^{n} r_{mj}] \quad \cdots & [\bigvee_{j=1}^{n} r_{mj}, 1] \end{bmatrix}$$

is the solution of the equation(2)

2, the smallest solution of the equation (2) is

$$(X) = \begin{bmatrix} \bigvee_{j=1}^{n} r_{1j} & & & 0 \\ & \bigvee_{j=1}^{n} r_{2j} & & \\ & & \ddots & \\ 0 & & & \bigvee_{j=1}^{n} r_{mj} \end{bmatrix}$$

Proof: first tesify 1 ,take any line i in (X)

$$([0, \bigwedge_{j=1}^{n} r_{ij}] \cdots [\bigvee_{j=1}^{n} r_{ij}, 1] \cdots [0, \bigwedge_{j=1}^{n} r_{ij}])$$

and column j in (R)$[r_{1j}, r_{2j}, \ldots, r_{mj}]^T$; j=1,2,...n, according to the combining method ,then have

$$([0, \bigwedge_{j=1}^{n} r_{ij}] \cdots [\bigvee_{j=1}^{n} r_{ij}, 1] \cdots [0, \bigwedge_{j=1}^{n} r_{ij}]) \circ$$

$$\circ [r_{1j}, r_{2j}, \ldots, r_{mj}]^T = r_{ij}$$

from the arbitrary property of i and j ,it is known

$$(X) \circ (R) = (R)$$

that (X)is the solution of (2) and since any element in (x)'belongs to the corresponding field of (x),thus (x)' being obviously the solution of the equation(2).

Further prove (x)'is the smallest solution.Besides some on the diagonal,the others have been the smallest if there exists a fuzzy matrix

$$(X)^{*}=\begin{bmatrix} d_{11} & & & 0 \\ & d22 & & \\ & & \ddots & \\ 0 & & & d_{mm} \end{bmatrix}$$

In it there at least exists a element-d_{ii}, which is smaller thancorresponding element $\overset{n}{\underset{j=1}{\vee}} r_{ij}$, in (X)',then there must exist a j and make $d_{ii}<d_{ij}$,in $(X)^{*}$take out line i

$$(0 \quad 0 \quad \ldots d_{ii} \quad 0 \ldots 0)$$

and take out colume j in (R) $[r_{1j}, r_{2j}, \cdots r_{mj}]^{T}$

at this time

$$(0\ 0 \ldots d_{ii} 0 \ldots 0)\circ\begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{mj} \end{bmatrix}=d_{ii}<r_{ij}$$

therefore have

$$(X)^{*}\circ(R)=(R)$$

means that $(X)^{*}$is not the solution of (2),thus having proved (X)' is the smallest solution of (2),also because to any given

$$R=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} \qquad (3)$$

(X),(X)' can be identified all the time,This indicates that equation (2) has the solution of identity.

In a lot of practical problems,solutions are often required

$$(R)\circ(X)=(S)$$

to solve this kind of fuzzy equation. Now let's discuss when (R)=(S), that is equation-solving of the equation like this.

$$(R)\cdot(X)=(R)$$

Suppose (R) formula(3) then the equation is

$$\begin{bmatrix} r_{11} & r_{12} & \cdots r_{1n} \\ r_{21} & r_{22} & \cdots r_{2n} \\ \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots r_{mn} \end{bmatrix} \cdot (X)= \begin{bmatrix} r_{11} & r_{12} & \cdots r_{1n} \\ r_{21} & r_{22} & \cdots r_{2n} \\ \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots r_{mn} \end{bmatrix} \qquad (4)$$

The proving method is the same as that of theorem 1, then we cantestify that equation (4) can satisfy the following theorems.

Theorem II.

1°,

$$(X)=\begin{bmatrix} [\bigvee_{i=1}^{m} r_{i1},1] & [0,\bigwedge_{i=1}^{m} r_{i2}] & \cdots & [0,\bigwedge_{i=1}^{m} r_{in}] \\ [0,\bigwedge_{i=1}^{m} r_{i1}] & [\bigvee_{i=1}^{m} r_{i2},1] & \cdots & [0,\bigwedge_{i=1}^{m} r_{in}] \\ \cdots & \cdots & & \cdots \\ [0,\bigwedge_{i=1}^{m} r_{i1}] & [0,\bigwedge_{i=1}^{m} r_{i2}] & \cdots & [\bigvee_{i=1}^{m} r_{in},1] \end{bmatrix}_{n\times n}$$

is the solution of equation(4)

2, The smallest solution of equation (4) is

$$(X)'=\begin{bmatrix} \bigvee_{i=1}^{m} r_{i1} & & & 0 \\ & \bigvee_{i=1}^{m} r_{i2} & & \\ & & \ddots & \\ 0 & & & \bigvee_{i=1}^{m} r_{in} \end{bmatrix}_{n\times n}$$

Appearently we have just given the form of the solution of the above special fuzzy equation, but in fact we have given the solution, too, because the corresponding elements to any fuzzy matrix (r_{ij}), $(X)'$ and (X) can be obtained directly according to the law of "$\wedge$" and "$\vee$".

II The solution of the general fuzzy equation

Now let's study the solution of the equation like this:

$$\begin{bmatrix} x_{11}x_{12} & \cdots x_{1k} \\ x_{21}x_{22} & \cdots x_{2k} \\ \cdots & \cdots \\ x_{m1}x_{m2} & \cdots x_{mk} \end{bmatrix} \circ \begin{bmatrix} r_{11}r_{12} & \cdots r_{1n} \\ r_{21}r_{22} & \cdots r_{2n} \\ \cdots & \cdots \\ r_{k1}r_{k2} & \cdots r_{km} \end{bmatrix} = \begin{bmatrix} s_{11}s_{12} & \cdots s_{1n} \\ s_{21}s_{22} & \cdots s_{2n} \\ \cdots & \cdots \\ s_{m1}s_{m2} & \cdots s_{mn} \end{bmatrix} \quad (5)$$

Since (5) can be always be changed into the solution of the following equations.

$$(x_{i1}x_{i2} \cdots x_{ik}) \circ \begin{bmatrix} r_{11}r_{12} & \cdots r_{1n} \\ r_{21}r_{22} & \cdots r_{2n} \\ \cdots & \cdots \\ r_{k1}r_{k2} & \cdots r_{kn} \end{bmatrix} = (s_{i1}s_{i2} \cdots s_{in})$$

$j=1,2,\ \ldots,\ m$. The following is the discussion just according to this equation.

$$(x_1x_2 \cdots x_k) \circ \begin{bmatrix} r_{11}r_{12} & \cdots r_{1n} \\ r_{21}r_{22} & \cdots r_{2n} \\ \cdots & \cdots \\ r_{k1}r_{k2} & \cdots r_{kn} \end{bmatrix} = (s_1s_2 \cdots s_n) \quad (6)$$

In accordance with relational combining law, it is easy to know that the necessary conditions of the solution of equation (6) is $\max(r_{ij}) \geqslant s_j$. $j=1,2,\ \ldots\ k_i; j=1,2,\ \ldots,n$
In the following discussion, we ,suppose the equation satis- this condition.

First difine a few operation.

(一) U_c operation

Let A.B be dimensional vector ofgeneral n composed by regarding some closed-intevals and points as elements, then

$$AU_cB = \begin{cases} A' . & \text{If } A' \text{ entirely includes into } B \ ; \\ B . & \text{If } B \text{ entirely includes into } A \ ; \\ A' \text{ or } B. & \text{If neither of them canbe included by each other.} \end{cases}$$

(二) δ operation

$$\begin{bmatrix} r_{1j} \\ r_{2j} \\ \\ r_{kj} \end{bmatrix} \xrightarrow{\delta} \begin{bmatrix} \bar{r}_{1j} \\ \bar{r}_{2j} \\ \\ \bar{r}_{kj} \end{bmatrix}$$

in it

$$r_{ij} \longrightarrow \bar{r}_{ij} = \begin{cases} s_i \,, & r_{ij} > s_i \,; \\ [s_j\,,\,1]\,, & r_{ij} = s_j \,; \\ [o,1]\,, & r_{ij} < s_j \,. \end{cases}$$

$i=1,2,\ldots k$; $j=1,2,\ldots n$. knows from δ operation ,$\bar{r}_{ij}$ must be one of s_j, $[s_j,\ 1]$, $[o,1]$.

(=) Γoperation and Γ^*operation

Definition:

$$\bar{r}_{ij} \xrightarrow{\Gamma} \bar{\bar{r}}_{ij} = \begin{cases} [o,s_j]\,, & r_{ij}=s_j \,; \\ [o,1]\,, & \bar{r}_{ij}=[s_i,1]\,; \\ [o,1]\,, & r_{ij}=[o,1] \,. \end{cases}$$

To all the appeared s_j or$[s_j,1]$of $[r_{1j},r_{2j},\ldots r_{kj}]^T$respectively fix once,and making Γoperation of the rest elements,and further making"$\mathcal{U}_c$"operation of all the results,and all the operations here is written as Γ^* .

For example.

$$\begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{kj} \end{bmatrix} \xrightarrow{\delta} \begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{kj} \end{bmatrix} \xrightarrow{\Gamma^*} \bigcup_{t=1}^{k'} \begin{bmatrix} \bar{\bar{r}}_{1j}\ (t) \\ \bar{\bar{r}}_{2j}\ (t) \\ \vdots \\ \bar{\bar{r}}_{kj}\ (t) \end{bmatrix}$$

Here k'represents the appeared s_j or the total number of $[s_j,\ 1]$ after δ operation of $[r_{1j},r_{2j},\ldots r_{kj}]^T$

Now let's give the solution of (6),write out the matrix corresponding to (6)

$$\begin{array}{ccc} s_1 & s_2 \cdots & s_n \end{array}$$
$$\begin{bmatrix} r_{11} & r_{12}\cdots & r_{1n} \\ r_{21} & r_{22}\cdots & r_{2n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ r_{k1} & r_{k2}\cdots & r_{kn} \end{bmatrix}$$

make operation of $j=1,2,\ldots n$

$$\begin{array}{c} S_j \\ \begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{kj} \end{bmatrix} \end{array} \xrightarrow{\delta} \begin{bmatrix} \bar{r}_{1j} \\ \bar{r}_{2j} \\ \vdots \\ \bar{r}_{kj} \end{bmatrix} \xrightarrow{\Gamma^*} \bigcup_{t=1}^{k'} \begin{bmatrix} \bar{\bar{r}}_{1j}(t) \\ \bar{\bar{r}}_{2j}(t) \\ \vdots \\ \bar{\bar{r}}_{kj}(t) \end{bmatrix}$$

Here $k' \leqslant n$, t is a fixed row coordinate of $[\bar{r}_{1j}, \bar{r}_{2j}, \ldots \bar{r}_{kj}]^T$ we have

Theorem III.

From $\bigcup_{t=1}^{k'}[\bar{\bar{r}}_{1j}(t), \bar{\bar{r}}_{2j}(t), \ldots, \bar{\bar{r}}_{kj}(t)]^T$,take out all the possible combination of n elements different from j(j=1,2, ...n),and respectively operate "$\wedge$" and make operation of all the results,the final result just being the solution of (6)

For example . The solving of the equation

$$(x_1 x_2 x_3 x_4) \circ \begin{bmatrix} 0.8 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.7 \\ 0.7 & 0.3 & 0.6 \\ 0.5 & 0.7 & 0.9 \end{bmatrix} = (0.8 \quad 0.6 \quad 0.6)$$

To explain the given solution,we carry it out in a few steps as follow.

(1)First solve the equation

$$(x_1 \ x_2 \ x_3 \ x_4) \circ \begin{bmatrix} 0.8 \\ 0.6 \\ 0.7 \\ 0.5 \end{bmatrix} = (0.8)$$

$$\begin{array}{c} 0.8 \\ \begin{bmatrix} 0.8 \\ 0.6 \\ 0.7 \\ 0.5 \end{bmatrix} \end{array} \xrightarrow{\delta} \begin{bmatrix} [0.8, & 1] \\ [0, & 1] \\ [0, & 1] \\ [0, & 1] \end{bmatrix} (1) \xrightarrow{\Gamma^*} \begin{bmatrix} [0.8, 1] \\ [0, & 1] \\ [0, & 1] \\ [0, & 1] \end{bmatrix} (2)$$

(2)Solve the equation

$$(x_1 x_2 x_3 x_4) \circ \begin{bmatrix} 0.5 \\ 0.6 \\ 0.3 \\ 0.7 \end{bmatrix} = (0.6)$$

$$\begin{array}{c} 0.6 \\ \begin{bmatrix} 0.5 \\ 0.6 \\ 0.3 \\ 0.7 \end{bmatrix} \end{array} \xrightarrow{\delta} \begin{bmatrix} [0,1] \\ [0.6,1] \\ [0,1] \\ [0,6] \end{bmatrix} (3) \xrightarrow{\Gamma^*} \begin{bmatrix} [0, 1] \\ [0.6, 1] \\ [0, 1] \\ [0, 0.6] \end{bmatrix} (4) \quad \cup \quad \begin{bmatrix} [0,1] \\ [0,1] \\ [0,1] \\ [0.6] \end{bmatrix} (5)$$

(3) Solve the equation

$$(x_1 x_2 x_3 x_4) \circ \begin{bmatrix} 0.5 \\ 0.7 \\ 0.6 \\ 0.9 \end{bmatrix} = (0.6)$$

$$\begin{array}{c} 0.6 \\ \begin{bmatrix} 0.5 \\ 0.7 \\ 0.6 \\ 0.9 \end{bmatrix} \end{array} \xrightarrow{\delta} \begin{bmatrix} [0,1] \\ 0.6 \\ [0.6,1] \\ 0.6 \end{bmatrix} (6) \xrightarrow{\Gamma^*} \begin{bmatrix} [0,1] \\ 0.6 \\ [0,1] \\ [0,0.6] \end{bmatrix} \cup (7) \begin{bmatrix} [0,1] \\ [0,0.6] \\ [0.6,1] \\ [0,0.6] \end{bmatrix} (8) \cup \begin{bmatrix} [0,1] \\ [0,0.6] \\ [0,1] \\ 0.6 \end{bmatrix} (9)$$

Thus the solution of the original equation can be gotten in the following method

$$\begin{array}{ccc} 0.8 & 0.6 & 0.6 \\ \end{array}$$
$$\begin{bmatrix} 0.8 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.7 \\ 0.7 & 0.3 & 0.6 \\ 0.5 & 0.7 & 0.9 \end{bmatrix} \xrightarrow{\delta} ((1)(3)(6)) \xrightarrow{\Gamma^*} (2 \wedge 4 \wedge 7) \cup_c (2 \wedge 4 \wedge 8) \cup_c (2 \wedge 4 \wedge 9) \cup_c (2 \wedge 5 \wedge 7) \cup_c (2 \wedge 5 \wedge 8) \cup_c (2 \wedge 5 \wedge 9) = \begin{bmatrix} [0.8,1] \\ 0.6 \\ [0,1] \\ [0,0.6] \end{bmatrix} \cup_c \begin{bmatrix} [0.8,1] \\ [0,0.6] \\ [0,1] \\ 0.6 \end{bmatrix}$$

III The solution of the equation $(R) \circ (X) = (S)$

Generally speaking

$$\begin{bmatrix} r_{11} r_{12} \cdots r_{1m} \\ r_{21} r_{22} \cdots r_{2m} \\ \cdots \cdots \cdots \\ r_{n1} r_{n2} \cdots r_{nm} \end{bmatrix} \circ \begin{bmatrix} x_{11} x_{12} \cdots x_{1n} \\ x_{21} x_{22} \cdots x_{2n} \\ \cdots \cdots \cdots \\ x_{m1} x_{m2} \cdots x_{mn} \end{bmatrix} = \begin{bmatrix} s_{11} s_{12} \cdots s_{1n} \\ s_{21} s_{22} \cdots s_{2n} \\ \cdots \cdots \cdots \\ s_{n1} s_{n2} \cdots s_{nn} \end{bmatrix} \quad (7)$$

can always be converted into the problems of equation-solving of a few equation like this.

$$\begin{bmatrix} r_{11} r_{12} \cdots r_{1m} \\ r_{21} r_{22} \cdots r_{2m} \\ \cdots \cdots \cdots \\ r_{n1} r_{n2} \cdots r_{nm} \end{bmatrix} \circ \begin{bmatrix} x_{1j} \\ x_{2j} \\ \cdots \\ x_{mj} \end{bmatrix} = \begin{bmatrix} s_{1j} \\ s_{2j} \\ \cdots \\ s_{nj} \end{bmatrix} \quad (8)$$

$(j=1,2,\ldots,n)$. Consequently the solution required of (7) only need finding out the solution of (8), then ranging it into matrix according to the order of the magnitute of j. And formula (8) can be converted into the following form.

$$(x_{ij} x_{2j} \cdots x_{mj}) \circ \begin{bmatrix} r_{11} r_{21} \cdots r_{n1} \\ r_{12} r_{22} \cdots r_{n2} \\ \cdots \cdots \cdots \cdots \\ r_{1m} r_{2m} \cdots r_{nm} \end{bmatrix} = (s_{1j} s_{2j} \cdots s_{nj})$$

This equation is the form of equation(6),then can be solved in the following method.

For example. Equation

$$\begin{bmatrix} x_{11}x_{12}x_{13} \\ x_{21}x_{22}x_{23} \end{bmatrix} \circ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \\ 0 & 0.6 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.2 \end{bmatrix}$$

can be solved in the above method to be

$$\begin{Vmatrix} \begin{vmatrix} 0.2 \\ (0,0.2) \\ 0.4 \end{vmatrix} & U_c & \begin{vmatrix} (0,0.2) \\ 0.2 \\ 0.4 \end{vmatrix} \\ \begin{vmatrix} (0,0.2) \\ 0.2 \\ 0.4 \end{vmatrix} & U_c & \begin{vmatrix} 0.2 \\ (0,0.2) \\ 0.4 \end{vmatrix} \end{Vmatrix}$$

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