FUZZY COMPLEMENTATION REVISITED

Herbert TOTH

A-1140 Wien, Riedstraße 17-19, Austria

Abstract: This paper introduces new proposals for fuzzy subsetness and fuzzy complementation, respectively. It is shown that the new formalism does exhibit quite nice properties: e.g. the complementation paradox vanishes, and the complement of a fuzzy set to another (proper !) fuzzy set becomes available. More details about the intuitive background can be found in [3].

1. Some general criticisms

As the terminology itself strongly suggests, one would believe that 'fuzzy set theory' and 'degree of membership' may have to do something with either physical (a fuzzy set of tall men) or mental (the set of natural numbers) collections of things. On the other hand, by far the most people fuzzy sets clearly are understood and used as a mathematical tool for modeling vague concepts. Thus, a logical component, at once, has one of the leading parts in the play, since concepts or notions are usually formalized by predicates.

Therefore, it does not make me wonder, that set theoretical and logical concepts, operations, and methods have freely been mixed together. As a consequence thereof, during the last few years a considerable number of critical papers have appeared (for more details see [3]). But, and this is the main idea behind my criticisms, we have to distinguish carefully between the syntactical/logical and the semantical/ontological dimension in the theory of fuzzy sets. In other words, we must always be aware of what we really have in mind: to handle/manipulate (collections of) objects themselves (semantical dimension), or to handle/maipulate valuations (over collections) of objects (syntactical dimension), whatever meaning we associate to these valuations, like e.g. utilities, probabilities, 'membership degrees' etc.

As is well known, the two quite different operations of logical negation and set theoretical complementation arithmetically coincide within the classical theories via the characteristic functions of sets and predicates, respectively. And as a consequence of this collapsing, we have only one generalized notion of fuzzy negation or complementation, respectively; therefore it is not astonishing at all that we get such intuitively curious results as have been criticized in [1], §4, in case $n(x):=1-x$ is interpreted as a generalization of a set theoretical operation.

Kabala's and Wroczinski's argumentation, adapted to our notations and examples, is as follows: Let S denote the fuzzy set of 'reals much greater than 5' (Figure 1a), and let R denote the fuzzy event 'choosing a real number $x\in[8,13]$ much greater than 5' (Figure 1b). In this situation $\bar{R}_S$ (Figure 1c) seems to fit more naturally into the conception of a complement than does the usual $\bar{R}$ (Figure 1d).

In order to avoid this deficiency, the notion of f-complementation is introduced and is shown to be both intuitively and formally much more suited as a fuzzy set-theoretical analogon of classical complementation.

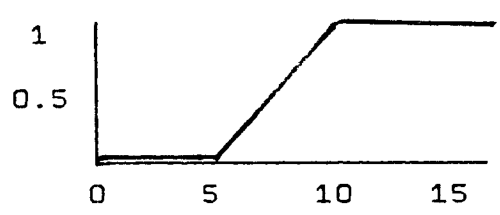

Fig.1a the fuzzy set S of all reals "much greater than 5"

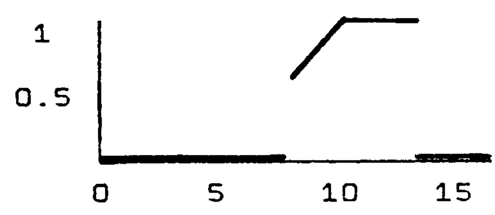

Fig.1b the fuzzy set R of reals "much greater than 5 and between 8 and 13"

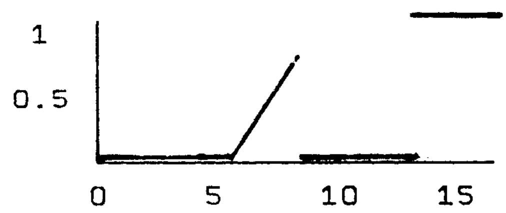

Fig.1c $\bar{R}_S$, the f-complement of fuzzy set R with respect to the proper fuzzy set S

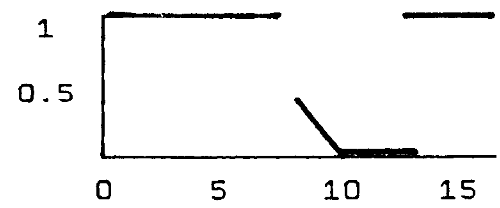

Fig.1d $\bar{R}$, the usual fuzzy compelement of R (i.e. with respect to the crisp universe $\mathbb{R}_0$)

Let me start with explaining the intuitive background for the following formal considerations: One of the basic and fundamental ideas of f-set theory is the assumption that the only kind of fuzzy sub-sets are the so-called standard subsets ([3]), i.e. that it is not - as is usually proposed - the ≤-relation between 'membership degrees', which is the essential characteristic of being a subset; but instead, it is the property of not being a member in the subset at all or of being a member "to the same degree" as in the underlying (super)set.

I believe to give sufficient support for this with the following argumentations: Considering the concept of extension and its possible connections to the notion of a fuzzy set, viz. to fuzzy sets in the sense of f-set theory (what else a sub-extension should be than an extension corresponding to a standard subset): There is, in my opinion, no doubt that speaking of a subset of A in the classical theory means to speak about a part of A. Generalizing this situation to the usual definition of a fuzzy subset, one should be able to answer the following question: "Which part of fuzzy set A of "reals much greater than 5" (Fig. 2a) corresponds to its fuzzy subset B of those reals, which are "very much greater than 5" (Figure 2b) ? With the given representation I feel the answer should be that both sets (as collections) are the same!

Thus, we have $B \subset A$, i.e B is a proper 'fuzzy subset' of A, but B is not part of A, which seems to me to be a rather strange result. Of course B is not a subextension of A either: B is an extension different from A.

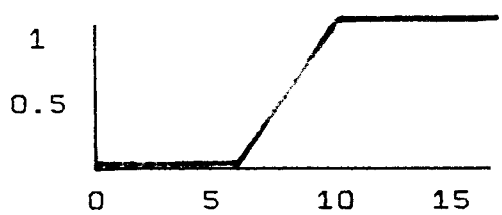

Fig.2a the fuzzy set A of all reals "much greater than 5"

Fig.2b the fuzzy set B of reals "very much greater than 5"

2. Basic Formalism

Definition 2.1 Let $U=\{u_1, u_2, \ldots, u_n\}$ be a finite set, called universe of discourse (universe, basic space, underlying universe), and let $V=\{v_0, v_1, \ldots, v_m\}$ be a finite partially ordered set, called valuation space. The set of all mappings from U to V is called the f-space generated by V and U, and will be denoted by V^U. As shorthand we will also use the term f-space V^U.

For sake of mathematical simplicity, as a first approach we will assume V to be a chain throughout the paper, i.e. $0:=v_0<v_1< \ldots < v_n =: 1$, $n \geq 1$. As usually, we define $\leq$ as the reflexive version of $<$, and let $\wedge$ and $\vee$ denote the meet and join operation in V, respectively. But assuming V to be a chain, i.e. $\leq$ to be a total order on V, has non-mathematical reasons too - and they are the more important ones from the viewpoint of f-set theory: If we had not made this assumption, there would exist valuations $v,w \in V$ which are $\leq$-incomparable. Such a mathematical model would correspond to a situation, where we classify two objects as being unequal but are unable to say which of them does exhibit a certain property to a greater extent. This I regard to be unnatural, i.e. not suitably describing everyday life, because we are in fact choosing or avoiding one of the two alternatives if necessary.

Definition 2.2 An f-set is an element of an f-space V^U. Let A and B denote two f-sets in the f-space V^U. A is called f-subset of B, denoted $A \sqsubseteq B$, iff $A(x)=0$ or $A(x)=B(x)>0$. The empty f-set $\emptyset$ is defined as follows: $\forall u \in U$: $\emptyset(u)=0$. (Note that we have used the term "standard subset" instead of "f-subset" up to now.)

For sake of notational convenience let us assume the basic space U as being arranged in an arbitrary, but fixed order. Thus, instead of of the tedious and longer $\{\langle x_1, v_A(x_1)\rangle, \ldots, \langle x_n, v_A(x_n)\rangle\}$ we simply will write $v_1v_2\ldots v_n$ to specify an f-set A over a universe U, where $|U|=n$ and $v_i := v_A(x_i)$. Obviously the valuation over U with respect to A is equally well characterized by each of the two ways of notation. This will presently become clear from the following

Example 1: We let $U=\{u_1, u_2\}$,i.e. $|U|=2$, and $V=\{0,a,1\}$, i.e. $|V|=3$, with $0<a<1$. We then get the following space of f-sets where the lines shall indicate the relation of being an f-subset. Note that lines arising from reflexivity or transitivity have not been inserted.

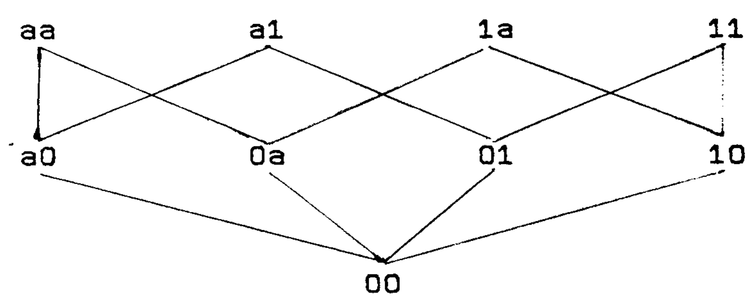

Fig. 3 The f-space V^U, generated by $U=\{u_1, u_2\}$ and $V=\{0,a,1\}$, with $0<a<1$.

Let $supp_A$:= $\{x \in U | A(x)>0\}$; equating $supp_A$ with $\{x \in U | P_A(x)\}$ is a remarkable consequence of our operational approach (see also [3]): There are only the two alternatives for a (physical or mental) object of being or not being member of an extension (set) $\mathcal{E}$, since we are grouping things together in a collection, according to whether they have or have not a certain property $\mathcal{P}$. That is to say: Only within the the support $supp_A$ things can be ordered with respect to the degree to which they do exhibit a property $\mathcal{P}$. In f-set theory the resulting order is modeled by the valuation $A:U \to V$.

The following propositions present a summary of the most fundamental properties of f-set theory. Their proofs generally are straightforward and therefore ommitted in most cases.

Proposition 3.1 $\sqsubseteq$ is an order on V^U, i.e. reflexive, transitive and antisymmetric.

Proof. (a) $\forall X \in V^U$: $X \sqsubseteq X$ by definition 2.2

(b) $A \sqsubseteq B \wedge B \sqsubseteq C$
$\to [A(x)=0 \vee A(x)=B(x)>0] \wedge [B(x)=0 \vee B(x)=C(x)>0]$
$\to A(x)=0 \vee A(x)=B(x)=C(x)>0$
$\to A \sqsubseteq C$

(c) $A \sqsubseteq B \wedge B \sqsubseteq A$
$\to [A(x)=0 \vee A(x)=B(x)>0] \wedge [B(x)=0 \vee B(x)=A(x)>0]$
$\to A(x)=B(x)=0 \vee A(x)=B(x)>0$
$\to A=B$

Proposition 3.2 Let $A,B \in V^U$; we define the f-set C by

$$C(x) = \begin{cases} A(x) & \text{if } A(x)=B(x) \\ 0 & \text{otherwise} \end{cases}$$

; then C is the greatest lower bound of A and B, denoted by g.l.b.$\{A,B\}$.

Proof. We obviously have $C \in V^U$ and $C \sqsubseteq A \wedge C \sqsubseteq B$, i.e. C is a lower bound of A and B. If C' is another lower bound, then $C'(x)=0 \vee C'(x)=A(x)=B(x)>0$; thus $C'(x)>0 \to C'(x)=C(x)$, i.e. $C' \sqsubseteq C$.

Definition 3.3 $A \sqcap B$:= g.l.b.$\{A,B\}$ is called the f-intersection of A and B.

Lemma 3.4 (Monotonicity of $\sqcap$) $A \sqsubseteq B \to A \sqcap C \sqsubseteq B \sqcap C$

Note that $\sqcap$ is not a restriction of the usual $\cap$ (pointwise min-operation) to certain special pairs of f-sets: In example 1 given above we have e.g. $aa \sqcap 1a = 0a$!

Proposition 3.5 (Associativity of $\sqcap$) $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C)$

Corollary 3.6 $(V^U, \sqcap)$ is a $\sqcap$-semilattice.

It is clear from the above by induction that any finite collection $\mathcal{A} = \{A_i\}_{1 \leqslant i \leqslant n}$ of f-sets in an f-space V^U admits a greatest lower bound in V^U; we then write $\text{g.l.b.}\{A_i\}_{1 \leqslant i \leqslant n} = \text{g.l.b.}\mathcal{A} = \sqcap \mathcal{A} = \bigsqcap_{1 \leqslant i \leqslant n} A_i$

Proposition 3.7 Let $\mathcal{U}_{AB} = \{U_i\}_{1 \leqslant i \leqslant n}$ be the set of upper bounds of A and B. Let U_{AB} denote the greatest lower bound of $\mathcal{U}_{AB}$, i.e. $U_{AB} = \bigsqcap_i U_i$. Then U_{AB} is the least upper bound of A and B, and will be denoted by $\text{l.u.b.}\{A,B\}$.

Definition 3.8 Let $\emptyset \neq \mathcal{U}_{AB} = \{U_i\}_{1 \leqslant i \leqslant n}$ be the set of upper bounds of A and B; then we define $A \sqcup B := \text{g.l.b.}\mathcal{U}_{AB}$, called the f-union of A and B.

Corollary 3.9 $A \sqsubseteq B \to A \sqcap B = A \wedge A \sqcup B = B$

We thus obviously get $\emptyset \sqcap X = \emptyset$ and $\emptyset \sqcup X = X$, i.e. $\emptyset$ as usually acts as a zero and one with regard to $\sqcap$ and $\sqcup$, respectively. By transitivity of $\sqsubseteq$ it follows that $A \sqcap B \sqsubseteq A \sqcup B$, if the latter one exists in V^U.

<u>Lemma 3.10</u> Let $A,B\in V^U$. If there exists an f-set $C\in V^U$ defined by (2) then $C = A\sqcup B$.

$$C(x) := \begin{cases} A(x) & \text{if } B(x)=0, \\ B(x) & \text{if } A(x)=0, \\ A(x) & \text{if } A(x)=B(x)>0 \end{cases} \tag{1}$$

(Note that $0<A(x)\neq B(x)>0$ implies $U_{AB}=\emptyset$!)

$\sqcup$ is commutative and idempotent, as is easily seen from Lemma 3.10.

Furthermore we see that C equals the usual $A\cup B$ (pointwise max-operation) if it exists in V^U according to (1).

R e m a r k: The following results only hold if all the involved unions exist in the f-space under consideration ! (Indicated by the existential quantifier $\exists$)

<u>Lemma 3.11</u> (Monotonicity of $\sqcup$) $A\sqsubseteq B \rightarrow A\sqcup C \;{}_{\exists}{\sqsubseteq}_{\exists}\; B\sqcup C$, if the corresponding f-sets are $\sqsubseteq$-comparable.

<u>Proposition 3.12</u> $(A\sqcup B)\sqcup C \;{}_{\exists}{=}_{\exists}\; A\sqcup(B\sqcup C)$

<u>Proposition 3.13</u> 1) $A\sqcap(A\sqcup B) \;{}_{\exists}{=}\; A\sqcup(A\sqcap B) = A$

2) $(A\sqcap B)\sqcup B \;{=}_{\exists}\; (A\sqcup B)\sqcap B = A$

<u>Proposition 3.14</u> 1) $A\sqcap(B\sqcup C) \;{}_{\exists}{=}_{\exists}\; (A\sqcap B)\sqcup(A\sqcap C)$

2) $A\sqcup(B\sqcap C) \;{}_{\exists}{=}_{\exists}\; (A\sqcup B)\sqcap(A\sqcup C)$

<u>Proposition 3.15</u> $A\sqcap(B\sqcup C) \;{}_{\exists}{\sqsubseteq}_{\exists}\; (A\sqcap B)\sqcup C$

3. f-complementation

<u>Definition 4.1</u> $A\in V^U$ is an <u>f-universe</u> in V^U if $supp_A=U$; an f-universe A in V^U is called <u>B-universe</u> and is denoted by U_B, if $B\sqsubseteq A$.

<u>Definition 4.2</u> Let $A\in V^U$; $B\in V^U$ is called <u>f-complement of A with valuation v</u>, $v\neq 0$, iff

$$B(x) = \begin{cases} 0 & A(x)>0 \\ v & A(x)=0 \end{cases} \tag{2}$$

(For sake of convenience, B sometimes will simply be called v-complement of A, and is denoted ${}_v\bar{A}$.)

Let me give some arguments for the plausibility of this definition: Interpreting the degree 0 as usual as the sign for denoting the fact that the corresponding element is not member of a set A under consideration, "the" complement of A in classical set theory (which in fact is with respect to the universe V) arises by putting together all those elements of the underlying universe, which are not members of A, into another set $\bar{A}$. From our viewpoint of fuzzy set theory we do not know anything about an order of the elements within $\bar{A}$, and have thus arrived at the classical notion of a set as an "unordered collection". The problems are due to the very special nature of the valuation space **2**: 1 is the only valuation different from 0, and hence we of course get $\bar{A}(x)=1$ for all $x\in\bar{A}$. But if we use a valuation space V with $|V|>2$ we have more possibilities for constant valuations indicating membership in $\bar{A}$, viz. the $v_1=|V-\{0\}|$ different positive values > 0.

As a conclusion we must say that in f-set theory something like "the" complement of an f-set does not ad hoc exist; thus we have to further specify what is meant by 'complement of A', either as we have done in definition 4.2 or by using a reference set to obtain the 'complement of A with respect to B':

<u>Definition 4.3</u> Let $A\sqsubseteq B$; the <u>f-complement of A with respect to B</u> is the f-set $\bar{A}_B$ given by

$$\bar{A}_B(x) = \begin{cases} 0 & A(x)>0 \\ B(x) & A(x)=0 \end{cases} \qquad (3)$$

The distinction of definitions 4.2 and 4.3 is not present in the classical theory, because the two concepts of complementation coincide due to the special valuation space **2**. They can be unified in f-set theory too (they do not collaps!), provided that we make a suitable generalization of the commonly used definition of pseudocomplement [2,4]:

A lattice L is called relatively pseudo-complemented, if for any pair of elements $a,b \in L$ there exists $b \rightarrow a := \text{l.u.b.}\{x \in L \mid b \wedge x \leq a\}$, called the pseudocomplement of b relative to a. Each relatively pseudo-complemented lattice has a unit element $1 := a \rightarrow a$, but does not need to have a zero element.

Returning to f-set theory we then have the following situation: Both in the general many-valued case and in the special two-valued case, we are working with a chain as valuation lattice (according to our assumption of section 2) $V := [0,1] = \{0=v_0<v_1< \ldots <v_n=1\}$, $n \geq 1$. We can easily get the following equations for the crisp case, by successively making the implicit conditions more explicit and precise (let $a,b,v \in V$):

$$b \rightarrow a := \text{l.u.b.}\{v \mid b \wedge v \leq a\} = \sup\{v \in V \mid b \wedge v \leq a\} = \sup\{0 \leq v \leq 1 \mid b \wedge v \leq a\};$$

turning over with the notation to valuations of f-sets we further get:

$$B_A(x) = \sup\{0 \leq v \leq 1 \mid B(x) \wedge v \leq 0\} = \sup\{0 \leq v \leq A(x) \mid B(x) \wedge v \leq 0\}.$$

We thus immediately see, that the crucial point is the choice of the upper bound of the interval for v to vary in. In the classical two-valued theory, this interval was bounded from above by 1 as the only "positive" value different from 0, for every $x \in U$. This again is an example for collapsing: because there simply is no other choice, it seems as if there would be a global upper bound for v, independent of x. But generalizing the formalism in the above way by introducing local or individual upper= bounds for the interval in which v may assume its value, we immediately get formally consistent and plausible results covering all the new situations arising in f-set theory as a many-valued set theory.

The unified formalism results from the above considerations as follows:

$$\bar{A}_B(x) := A(x) \rightarrow_B 0 := \sup\{v \in [0, B(x)] \mid A(x) \wedge v \leq 0\} \qquad (4)$$

$$_d\bar{A}(x) := A(x)\,_d\!\rightarrow 0 := \sup\{v \in [0, d] \mid A(x) \wedge v \leq 0\} \qquad (5)$$

Note that (5) is of course a special case of (4): $_d\bar{A}(x) = \bar{A}_{_dU}(x)$, where $\forall x \in U$: $_dU(x) = d$.

We now introduce the following notations: $V_1 := V - \{0\}$, and

$_v\mathcal{Cst}(V,U) := \{A \in V^U \mid A(x) = 0 \vee A(x) = v \in V_1\}$,

$\mathcal{Cst}(V,U) := \bigcup_{v \in V_1} {_v\mathcal{Cst}}(V,U)$.

If U and V are clear from the context or do not need to be further specified (which will usually be the case in this paper) then we will shortly write $_v\mathcal{Cst}$ and $\mathcal{Cst}$, respectively. f-sets in $_v\mathcal{Cst}$ or $\mathcal{Cst}$ will be called v-constant or constant, respectively. A v-constant f-universe is called v-universe, for $v \in V_1$, and is denoted by $_vU$, i.e. $\forall x \in U$: $_vU(x) = v$.

Note that $\forall v \in V_1$: $\emptyset \in {_v\mathcal{Cst}}$; thus, $_v\mathcal{Cst}$ may well be denoted by $\mathcal{P}_v(U)$ in order to emphasize its analogy to the classical notion of a power set $\mathcal{P}(U)$ of U. Indeed, $\mathcal{B}_v := (\mathcal{P}_v(U), \sqcup, \sqcap, {_v}\bar{\ })$ forms a Boolean algebra which is isomorphic to $\mathbf{2}^U$; generally we have $\forall v,w \in V_1$: $\mathcal{B}_v \triangleq \mathcal{B}_w \triangleq \mathbf{2}^U$.

The following propositions present the key results of our new operational approach to fuzzy set theory:

<u>Proposition 4.4</u> 1) $\bar{A}_A = \emptyset$, $\overline{(\bar{B}_A)}_B = B$

2) $\bar{A}_B \sqsubseteq B$

3) $\forall v,w \in V_1$: supp $_w\overline{(_v\bar{A})} = \mathrm{supp}_A$

4) $A \in {_v\mathcal{Cst}} \longleftrightarrow \forall w \in V_1$: $_v\overline{(_w\bar{A})} = A$

Proof. We show only the validity of 2):

$$(B(x) \to_A 0) \qquad \to_B 0$$

$$= \begin{cases} A(x) & \text{if } B(x)=0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} B(x) & \text{if } B(x)\to_A 0=0 & \text{iff } B(x)>0 \\ 0 & \text{otherwise} & \text{iff } B(x)=0 \end{cases}$$

Proposition 4.5 1) $\forall v\in V_1$: $A\sqcap_v \bar{A} = \emptyset$

2) $\forall v\in V_1$: $A\sqcup_v \bar{A}$ is an A-universe.

3) $\forall v\in V_1$: $A\sqsubseteq B \to {}_v\bar{B}\sqsubseteq_v \bar{A}$

4) $\forall C\in V^U$: $A\sqsubseteq B \to \bar{B}_C\sqsubseteq \bar{A}_C$

Proposition 4.6 1) ${}_v\overline{(A\sqcap B)} \sqsupseteq {}_v\bar{A}\sqcup_v\bar{B}$

2) ${}_v\overline{(A\sqcup B)} \;\sqsupseteq\!= {}_v\bar{A}\sqcap_v\bar{B}$

References

[1] KABALA,Z./WROCZINSKI,I.: A short notice on Bayes' rule and related topics. BUSEFAL 18 (1984), 92-99.

[2] RASIOWA,M.: An algebraic approach to non-classical logic (North Holland, Amsterdam, 1974).

[3] TOTH,H.: From fuzzy-set theory to fuzzy set-theory: Some critical remarks on existing concepts. Fuzzy Sets and Systems, to appear.

[4] WEBER,S.: A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms. Fuzzy Sets and Systems 11 (1983), 115-134.