THE EXTENSION OF THE METHOD FOR FINDING TRANSMISSIVE CLOSURE FOR R RELATION ABOUT WARSHALL

Zhu Jisheng (Hebei Teachers' University),
Wang Qingyin , Tian Zukai , Coal Mining & Civil Engineering College of Hebei, Handan, Hebei Prov.
People's Republic of China

ABSTRACT: In this paper, we extend transmissive and closure system for finding common relation R of WARSHALL to the transmissive and closure system for finding fuzzy relation R.

KEYWORDS: Fuzzy Relation, Fuzzy Matrix, Ttansmissive Closure

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The transmissive and closure of fuzzy relation plays an important role both in theory and application. T.C. Dunn, on the basis of transmissive and closure R* about n-th-order fuzzy similar matrix R, put forward the following theorem:

Any $K \geq n$, there is $R^*=R^k$.

According to this theorem, R* can be obtained by "SQUARING", $R \to R^2 \to R^4 \to \ldots \to R^{2k} \to \ldots$

Through finitive calculation, R* is obtained, but in the case of finding R transmissive and closure when the R-th-order is larger, the calculation is trivil and tedious, this paper extends common transmissive and closure system of WARSHALL to finding the transmissive and closure of fuzzy relation, thus above mentioned disadvantage is not only avoided, but simultaneously it provides an general way for finding transmissive and closure fuzzy relation R.

Suppose $X=\{x_1, x_2, \ldots, x_n\}$, R is the fuzzy relation in X. Symbol $M_{n\times n}$ shows all $n\times n$ fuzzy matrix set. Let $R \in M_{n\times n}$, written $R=(R(i,j))$, $R(i,j) \in [0,1]$

Where, R(i,j) shows i line j row of Rth element.

DEFINITION Let $R \in M_{n\times n}$, the elements in R, according to their value, are defined respectively maximal element, 1st maximal element, 2nd maximal element, ... minimal element.

Suppose $R \in M_{n\times n}$, R constructs a new matrix $R^* \in M_{n\times n}$, the method is as follows:

(1) Let $R(j_o,i_o)$ be the maximal element in R, rewrite the element in line j_o of R respectively into

$(R(j_o,k) \vee R(i_o,k)) \wedge R(j_o,i_o)$, $K \in \{1,2, \ldots, n\}$.

Again let $R(j_o,i_o)$ be the element $R^*(j_o,i_o)$ of R^*, after change the element of R by the way mentioned above of the maximal element of R, the obtained new matrix is written R_o.

(2) Let $R_o(j_1,i_1)$ be the first maximal element of R_o, if the j_1 line element $R_o(j_1,k)$ of R_o satisfies $R_o(j_1,k) < R_o(j_1,i_1)$, rewrite respectively as

$(R_o(j_1,k) \vee R_o(i_1,k)) \wedge R_o(j_1,i_1)$, $K \in \{1,2,\ldots, n\}$

Again let $R_o(j_1,i_1)$ be the element $R^*(j_1,i_1)$ of R^*, after the said change, the changed matrix in R_o is written R_1.

(3) Let $R(j_2,i_2)$ be the 2nd maximal element of R_1, if the j_2 line element $R_1(j_2,k)$ in R_1 satisfies $R_1(j_2,k) < R_1(j_2,i_2)$, it is written respectively

$(R_1(j_2,k) \vee R_1(i_2,k)) \wedge R_1(j_2,i_2)$, $K \in \{1,2, \ldots, n\}$.

Again let $R_1(j_2,i_2)$ be the element $R^*(j_2,i_2)$ in R^*, after the said change, the changed matrix in R is written R_2.

(4) Similarly, construct other elements in R^* in the way listed in (1),(2),(3), for there is only different finitive element in R, finally the complete elements in R^* can therefore be constructed.

THEOREM The constructed R^* by R with above mentioned way, there is bound to be $R^*=t(R)$.

PROOF: From the construction R^*, it is known $R \subseteq R^*$.

In the following we shall prove R^* is transmissive, take $i_o \in \{1,2,\ldots, n\}$, suppose the elements $R^*(i_o,j_o)$ in R^* is the maximal element in line i_o, thus there is $R^{*2}(i_o,j_o) \leq R^*(i_o,j_o)$.

Suppose element $R^*(i_o,j_1)$ is the 1st maximal element in i_o line R^*.

$$R^{*2}(i_o,j\ )=((R^*(i_o,1) \wedge R^*(1,j_1)) \vee \ldots \vee (R^*(i_o,j_o) \wedge R^*(j_o,j_1)) \vee \ldots \vee (R^*(i_o,n) \wedge R^*(n,j_1)) .$$

There exists $R^*(j_o,j_1) \leq R^*(i_o,j_1)$, otherwise

$R^*(j_o,j_1) > R^*(i_o,j_1)$, thus from the construct of R^*, it is known $R(j_o,j_1) > R(i_o,j_1)$, or there exists $j_s \in \{1,2,\ldots, n\}$, making $((R(j_s,j_1) \vee R(j_o,j_1)) \wedge R(j_o,j_s) > R^*(i_o,j_1)$ and

$R(j_o,j_1) < R(j_o,j_s)$. (1)

Noting again, $R(i_o,j_o)$ is the maximal element in line i_o , Thus $R(j_o,j_1) > R^*(i_o,j_1)$ is not possible.

From (1), $R(j_s,j_1) \wedge R(j_o,j_s) > R^*(i_o,j_1)$,

Hence, $R(j_s,j_1) > R^*(i_o,j_1)$ and $R(j_o,j_s) > R^*(i_o,j_1)$.

When $R(j_o,j_s) \geq R(i_o,j_o)$, from R^* construct, $R(j_s,j_1) \leq R^*(i_o,j_1)$, it's contradictory to $R(j_s,j_1) > R^*(i_o,j_1)$;

When $R(j_o,j_s) < R(i_o,j_o)$, and from $R(i_o,j_o)$ is the maximal element in line i_o, having $R^*(i_o,j_o) > R^*(i_o,j_s) > R^*(i_o,j_1)$ or $R^*(i_o,j_s) = R^*(i_o,j_o)$, the former is contradictory to $R^*(i_o,j_1)$ being the 1st maximal element in line i_o, the later infers $R(j_s,j_1) \leq R^*(i_o,j_1)$, this is contradictory to

$R(j_s,j_1) > R^*(i_o,j_1)$;

When $R(j_o,j_s) < R(i_o,j_o)$, and from $R(i_o,j_o)$ is the maximal element in line i_o, having $R^*(i_o,j_o) > R^*(i_o,j_s) > R^*(i_o,j_1)$ or $R^*(i_o,j_s) = R^*(i_o,j_o)$, the former is contradictory to $R^*(i_o,j_1)$ being the 1st maximal element in line i_o, the later infers $R^*(j_o,j_s) \leq R^*(i_o,j_1)$, thus $R^{*2}(i_o,j_1) \leq R^*(i_o,j_1)$.

Similarly, it can be proved that any th maximal element $R^*(i_o, j_l)$ in line R^* satisfies $R^{*2}(i_o,j_l) \leq R^*(i_o,j_l)$.

Because of arbitrariness of i_o, any $i,j \in \{1,2,\ldots, n\}$, have $R^{*2}(i,j) \leq R^*(i,j)$, therefore R^* is transmissive.

PROOF Again, if $R \leq R'$ and $R'^2 \leq R'$, there exists $R^* \leq R'$.

Let $R(j_o,i_o)$ be the maximal element in R, and for

$R(j_o,i_o) = R^*(j_o,i_o)$, hence $R^*(j_o,i_o) \leq R'(j_o,i_o)$.

Here $(R(j_o,k) \vee R(i_o,k)) \wedge R(j_o,i_o) = R(j_o,k) \vee R(i_o,k), k \in \{1,2,\ldots,n\}$.

If $R(j_o,k) \vee R(i_o,k) = R(j_o,k)$, for $R(j_o,k) \leq R'(j_o,k)$,

Hence $R(j_o,k) \vee R(i_o,k) \leq R'(j_o,k)$.

If $R(j_o,k) \vee R(i_o,k) = R(i_o,k)$, from the transmissiveness of of R', we have

$R'(j_o,k) \geq \bigvee_{l=1}^{n}(R'(j_o,l) \wedge R'(l,k)) \geq R'(j_o,i_o) \wedge R'(i_o,k) \geq$
$R'(j_o,i_o) \wedge R'(i_o,k) = R(i_o,k) = R(j_o,k) \vee R(i_o,k)$.

Therefore any $K \in \{1,2,\ldots,n\}$, there exists $R'(j_o,k) \geq R(j_o,k) \vee R(i_o,k)$, thus $R_o \subseteq R'$.

Suppose $R_o(j_1,i_1)$ is the 1st maximal element in R_o, then $R'(j_1,i_1) \geq R^*(j_1,i_1)$, if the line j_1 element $R_o(j_1,k)$ in R_o satisfies $R_o(j_1,k) < R_o(j_1,i_1)$, then they can be re-written respectively:

$(R_o(j_1,k) \vee R_o(i_1,k) \wedge R_o(j_1,i_1)$

Then $(R(j_1,k) \vee R_o(i_1,k)) \wedge R_o(j_1,i_1) = R_o(i_1,k)$ (1)

Or $(R(j_1,k) \vee R_o(i_1,k)) \wedge R_o(j_1,i_1) = R_o(j_1,i_1)$ (2)

Or $(R(j_1,k) \vee R_o(i_1,k)) \wedge R_o(j_1,i_1) = R_o(j_1,k)$ (3)

In the case of (1), we have

$$R'(j_1,k) \geq \bigvee_{l=1}^{n}(R'(j_1,l) \wedge R'(l,k))$$
$$\geq R'(j_1,i_1) \wedge R'(i_1,k)$$
$$\geq R_o(j_1,i_1) \wedge R_o(i_1,k) = R_o(i_1,k)$$

Hence $R'(j_1,k) \geq (R_o(j_1,k) \vee R_o(i_1,k) \wedge R_o(j_1,i_1)$.

Similarly in case (2), we have

$$R'(j_1,k) \geq \bigvee_{l=1}^{n} (R'(j_1,l) \wedge R'(l,k)$$
$$\geq (R_o(j_1,k) \vee R_o(i_1,k)) \wedge R_o(j_1,i_1).$$

In case (3), it is obvious

$$R'(j_1,k) \geq (R_o(j_1,k) \vee R_o(i_1,k)) \wedge R_o(j_1,i_1).$$

Similarly we can prove any $l \in \{o,1,2,\ldots,\}$ there exists $R_l \subseteq R'$, hence $R^* \subseteq R'$, therefore $R^* = t(R)$.

From the said theorem and R^* construct system, we obtain the method for $t(R)$ from R:

(1) Let new matrix $A=R$;

(2) Let $i=1$

(3) If in all j, $A(j,i)$=maximal element, and $k=1,2,\ldots,n$ and when $A(j,k) < A(j,i)$;

Let $A(j,k) = (A(j,k) \vee A(i,k)) \wedge A(j,i)$;

(4) i plus 1;

(5) If $i \leq n$, then transfer to step (3), otherwise it stops.

(6) In (3), in all j, if A(j,i) respectively equals to 1st maximal element, 2nd maximal element, ..., minimal element, and in turn repeat steps (3)— (5), we could obtain finally t(R).

EXAMPLE 1. Let

$$R=\begin{bmatrix}0.5 & 0.3 & 0.4 & 0.6\\ 0.2 & 0.1 & 0.8 & 0.2\\ 0.3 & 0.4 & 0.5 & 0.6\\ 0.7 & 0.1 & 0.4 & 0.2\end{bmatrix} \text{Find } t(R).$$

SOLUTION:

$$A = R = \begin{bmatrix}0.5 & 0.3 & 0.4 & 0.6\\ 0.2 & 0.1 & 0.8 & 0.2\\ 0.3 & 0.4 & 0.5 & 0.6\\ 0.7 & 0.1 & 0.4 & 0.2\end{bmatrix}$$

When i=3, A(2,3)=0.8 be the maximal element, then rewrite the second element as:

$(A(2,k) \vee A(3,k) \wedge A(2,3),\ K \in \{1,2,3,4\}$.

We obtain:

$$A=\begin{bmatrix}0.5 & 0.3 & 0.4 & 0.6\\ 0.3 & 0.4 & 0.8 & 0.6\\ 0.3 & 0.4 & 0.5 & 0.6\\ 0.7 & 0.1 & 0.4 & 0.2\end{bmatrix}$$

When i=1,A(4,1)=0.7 is the 1st maximal element, if $A(4,k) < A(4,1)$, rewrite the 4th maximal elements as:

$(A(4,k) \vee A(1,k)) \wedge A(4,1),\ K \in \{1,2,3,4\}$

We obtain

$$\begin{bmatrix}0.5 & 0.3 & 0.4 & 0.6\\ 0.3 & 0.4 & 0.8 & 0.6\\ 0.3 & 0.4 & 0.5 & 0.6\\ 0.7 & 0.3 & 0.4 & 0.6\end{bmatrix}$$

If i=4,A(1,4)=0.6, A(2,4)=0.6, A(3,4)=0.6, A(4,4)=0.6 be the 2nd maximal element, then respectively rewrite the 1st line, 2nd line, 3rd line, 4th line elements as

$(A(1,k) \vee A(4,k)) \wedge A(1,4)$, (when $A(1,k) < A(1,4)$);

$(A(2,k) \vee A(4,k)) \wedge A(2,4)$, (when $A(2,k) < A(2,4)$);

$(A(3,k) \vee A(4,k)) \wedge A(3,4)$, (when $A(3,k) < A(3,4)$);

$(A(4,k) \vee A(4,k)) \wedge A(4,4)$, (when $A(4,k) < A(4,4)$).

We obtain

$$\begin{bmatrix}0.6 & 0.3 & 0.4 & 0.6\\ 0.6 & 0.4 & 0.8 & 0.6\\ 0.6 & 0.4 & 0.5 & 0.6\\ 0.7 & 0.3 & 0.4 & 0.6\end{bmatrix}$$

If i=1, A(1,1)=0.6, A(2,1)=0.6, A(3,1)=0.6 be the 2nd maximal element;

i=3, A(3,5)=0.5, be the 3rd **maximal** element;

i=2, A(2,2)=0.4, A(3,2)=0.4 be the 4th maximal element, it's easy to know that the given value of A is constant.

i=3, A(1,3)=0.4, A(4,3)=0.4 be the 4th maximal element, when $A(1,k) < A(1,3)$, rewrite the 1st line elements as

$(A(1,k) \vee A(3,k)) \wedge A(1,3)$, when $A(4,k) < A(4,3)$, rewrite the 4th line elements as $(A(4,k) \vee A(3,k)) \wedge A(4,3)$,

Obtaining

$$R=\begin{bmatrix} 0.6 & 0.4 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 & 0.6 \\ 0.6 & 0.4 & 0.5 & 0.6 \\ 0.7 & 0.4 & 0.4 & 0.6 \end{bmatrix}, \text{Hence } t(R)=\begin{bmatrix} 0.6 & 0.4 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 & 0.6 \\ 0.6 & 0.4 & 0.5 & 0.6 \\ 0.7 & 0.4 & 0.4 & 0.6 \end{bmatrix}.$$

For a simpler calculation, the calculation process can be carried out in the same matrix, omit other writings or presentation only by **crossing** out other elements which should be changed, and replace the should be rewritten elements, we can therefore carry the following calculation from mentioned above:

$$\begin{bmatrix} \substack{0.6 \\ \cancel{0.5}} & \substack{0.4 \\ \cancel{0.3}} & 0.4 & 0.6 \\ \substack{0.6 \\ \cancel{0.3} \\ \cancel{0.2}} & \substack{0.4 \\ \cancel{0.1}} & 0.8 & \substack{0.6 \\ \cancel{0.2}} \\ \substack{0.6 \\ \cancel{0.3}} & 0.4 & 0.5 & 0.6 \\ 0.7 & \substack{0.4 \\ \cancel{0.3} \\ \cancel{0.1}} & 0.4 & \substack{0.6 \\ \cancel{0.2}} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 & 0.6 \\ 0.6 & 0.4 & 0.5 & 0.6 \\ 0.7 & 0.4 & 0.4 & 0.6 \end{bmatrix} = t(R)$$

When one finds the transmissive and closure system for the similar matrix of fuzzy, because of its symetricalness, it is only necessary to re-construct its diagonal upper elements, see

EXAMPLE 2. $\begin{bmatrix} 1 & 0.1 & 0.8 & 0.5 & 0.3 \\ 0.1 & 1 & 0.1 & 0.2 & 0.4 \\ 0.8 & 0.1 & 1 & 0.3 & 0.1 \\ 0.5 & 0.2 & 0.3 & 1 & 0.6 \\ 0.3 & 0.4 & 0.1 & 0.6 & 1 \end{bmatrix}$, find $t(R)$.

SOLUTION:

1	0.4 ~~0.2~~ ~~0.1~~	0.8	0.5	0.5 ~~0.3~~
0.1	1	0.4 ~~0.3~~ ~~0.1~~	0.4 ~~0.2~~	0.4
0.8	0.1	1	0.5 ~~0.3~~	0.5 ~~0.3~~ ~~0.1~~
0.5	0.2	0.3	1	0.6
0.3	0.4	0.1	0.6	1

Thus

$$t(R)=\begin{pmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{pmatrix}.$$

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Biblography

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