SOME PROPERTIES OF FUZZY EIGEN SET

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Abstract

In 1978, E.Sancher advanced Fuzzy eigen equation the maximal solution's existence and to find solutions method. This paper gave every Fuzzy eigen equation expression of all solutions when coefficient matrices are idempotent, or diagonally dominant, or transitive, and proved that solution set of Fuzzy eigen equation of general coefficient matrix must be contained in one easy found solution set of Fuzzy eigen equation that coefficient matrix is idempotent.

KEY WORDS: Fuzzy matrix, eigen set, idempotent matrix, transitive closure, diagonally dominant matrix.

Def. 1 If matrix $A=(a_{ij})_{m\times n}$, $\forall i,j, a_{ij}\in[0,1]$, then A is called Fuzzy matrix.

Now we define addition and multiplication between any two numbers in $[0,1]$.

$$a+b=\max(a,b) \qquad ab=\min(a,b)$$

Similarly, we can define matrices's addition and multiplication.

$1\times n$ Fuzzy matrix is called Fuzzy row vector.

Similarly we can define Fuzzy column vector.

Def. 2 All n-dimension vectors are written W. If S is subset of W, and for addition and multiplication closed, then S is called a subspace of W.

All $n\times n$ matrices are written $L^{n\times n}$.

Def. 3 Let $A\in L^{n\times n}$. If $A^2=A$, then A is called idempotent.

Def. 4 Let $A\in L^{n\times n}$. If $\forall i,j\in\{1,2,\ldots,n\}$, all have $a_{ij} \leqslant a_{ii}$, then A is called diagonally dominant.

Def. 5 Let $A \in L^{nxn}$. If there is a Fuzzy matrix A^* to satisfy:

(1) $A^{*2} \subset A^*$

(2) $A \subset A^*$

(3) $A \subset S$, $S^2 \subset S$, then $A^* \subset S$

then A is called thansitive closure, written $A^*=t(A)$.

Def. 6 Fuzzy equation AX=X is called Fuzzy eigen equation of matrix A, where A is a nxn Fuzzy matrix, X is a n-dimension Fuzzy column vector to waiting find.

We stipulate n-dimension column vector:

$O=(0,0,\ldots,0)^T$

is a ordinary solution of Fuzzy eigen equation AX=X.

As is know to all, solution set of Fuzzy eigen equation AX=X consists of a subspace of W. The maximal solution of Fuzzy eigen equation is existence and sole.

Let $A \in L^{nxn}$, written $A^1=A$, $A^2=AoA$, $A^3=AoA^2,\ldots,$ then $A,A^2,A^3,\ldots,$ are called matrix A's one order power, two order power, three order power,..., their eigen sets are respectively written $E^{(1)}$, $E^{(2)}$, $E^{(3)}$,...

In the general, eigen sets of A's any order power not necessarily are same, but it is obvious that:

Prop. 1 If $A \in L^{nxn}$, and A is idempotent, then eigen sets $E^{(1)}$, $E^{(2)}$, $E^{(3)}$,..., are same.

They are easy proved:

Prop. 2 If $A \in L^{nxn}$, and A is idempotent, then every column vector of $A=(a_{ij})$,

$(a_{1j},a_{2j},\ldots,a_{nj})^T$ $j=1,2,\ldots,n$ must be a solution of AX=X.

Prop. 3 If $A \in L^{nxn}$, and A is idempotent, then for any a n-dimension column vector $\overline{X}=(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_n)^T$, where $\overline{x}_i \in [0,1]$.

$A\overline{X}$ must be a solution of AX=X.

Theorem 1 If $A \in L^{nxn}$, and A is idempotent, then

$(\bigvee_{j=1}^{n} a_{1j}, \bigvee_{j=1}^{n} a_{2j}, \ldots, \bigvee_{j=1}^{n} a_{nj})^T$ is the maximal solution of AX=X, where V denotes taking maximum, and symbol $\wedge$ will denote taking minimum.

Proof It is obvious that $(\bigvee_{j=1}^{n} a_{1j}, \bigvee_{j=1}^{n} a_{2j}, \ldots, \bigvee_{j=1}^{n} a_{nj})^T$ is a solution of AX=X.

Let $\overline{X}=(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^T$ be any a solution of AX=X. For $\forall i \in \{1,2,\ldots,n\}$ there is $a_{ij} \wedge \overline{x}_j \leqslant \bigvee_{j=1}^{n} a_{ij}$. $j=1,2,\ldots,n$ Hence $\bigvee_{j=1}^{n} (a_{ij} \wedge \overline{x}_j) \leqslant \bigvee_{j=1}^{n} a_{ij}$, i.e. $\overline{x}_i \leqslant \bigvee_{j=1}^{n} a_{ij}$, so $(\bigvee_{j=1}^{n} a_{1j}, \bigvee_{j=1}^{n} a_{2j}, \ldots, \bigvee_{j=1}^{n} a_{nj})^T$ is the maximal solution of AX=X.

Theorem 2 If $A \in L^{n\times n}$, and A is idempotent, and subspace G is produced by column vector group of A, then G identical $E^{(1)}$.

Proof Any a element in G is similar in from

$$\overline{X} = \sum_{i=1}^{n} r_i (a_{1i}, a_{2i}, \ldots, a_{ni})^T \quad r_i \in [0,1]$$

but $\overline{X} = \bigvee_{i=1}^{n} (r_i \wedge a_{1i}, r_i \wedge a_{2i}, \ldots, r_i \wedge a_{ni})^T = A \circ (r_1, r_2, \ldots, r_n)^T$

from proposition 2, $A \circ (r_1, r_2, \ldots, r_n)^T$ is a solution of AX=X, hence $\overline{X} \in E^{(1)}$.

Now suppose $Y \in E^{(1)}$. $Y=(y_1, y_2, \ldots, y_n)^T$ $y_i \in [0,1]$ then $Y = AY = \bigvee_{i=1}^{n} y_i (a_{1i}, a_{2i}, \ldots, a_{ni})^T = \sum_{i=1}^{n} y_i (a_{1i}, a_{2i}, \ldots, a_{ni})^T$ so $Y \in G$.

Such as above proved G identical $E^{(1)}$.

Theorem 2 expound such fact: If A is idempotent, then solution set expression of AX=X can fully determine.

Theorem 3 If $A \in L^{n\times n}$, and A is diagonally dominant then any a solution of AX=X must be a solution of eigen equation $A^*X=X$, where $A^*=t(A)$.

Proof Let $\overline{X}$ be any a solution of AX=X, i.e. $A\overline{X}=\overline{X}$, then $A(A\overline{X})=A\overline{X}=\overline{X}$, hence $\overline{X}=A\overline{X}=A^2\overline{X}=\ldots=A^n\overline{X}$, but $A^*=A^n$, so $A^*\overline{X}=\overline{X}$.

It is easy proved:

Theorem 4 If $A\in L^{n\times n}$, and A is diagonally dominant, then any a solution X^* of $A^*X=X$, must be a solution of $AX=X$.

Deduction Let $A\in L^{n\times n}$, and A is diagonally dominant, then the maximal solution of $AX=X$ is $(a_{11},a_{22},\ldots,a_{nn})^T$.

Proof It is obvious that $A^*=A^n$ is idempotent. By [3] proposition III-2, $E^{(1)}$ and $E^{(2)}$ both have same maximal element. Hence the maximal solution of $AX=X$ is combining of all column vectors in A^*,

i.e. $(\bigvee_{j=1}^{n} a^*_{1j},\bigvee_{j=1}^{n} a^*_{2j},\ldots,\bigvee_{j=1}^{n} a^*_{nj})^T$ and i.e.

$(a_{11},a_{22},\ldots,a_{nn})^T$ (Because the maximal element of every row and every column in A must be the maxinal element of relevant row and column in A^*.)

Theorem 5 If $A\in L^{n\times n}$, and A is diagonally dominant, then eigen sets $E^{(1)},E^{(2)},E^{(3)},\ldots,E^{(n)},\ldots$ are the same.

Proof First, because $A^*=A^n=A^{n+1}=\ldots$, hence $E^{(n)}=E^{(n+1)}=E^{(n+2)}=\ldots$. From theorem 3 and 4 know $E^{(1)}=E^{(n)}$, by suppose A is diagonally dominant, can obtain $A\subset A^2\subset\ldots\subset A^n\subset\ldots$, hence natural number s of any satisifed $1<s<n$, all have $A\subset A^s\subset A^*$.

Now suppose $\overline{X}$ is any a solution of $AX=X$, then have $\overline{X}=A\overline{X}\subset A^s\overline{X}\subset A^*\overline{X}=\overline{X}$, hence $\overline{X}$ is a solution of $A^sX=X$.

Must exist natural number $m>n$, send s/m to be positive integer. By [3] proposition III-1 have $E^{(s)}\subset E^{(m)}$. But $E^{(m)}=E^{(n)}=E^{(1)}$, hence $E^{(s)}=E^{(1)}$, therefore $E^{(1)},E^{(2)},\ldots,E^{(n)},\ldots$ are the same.

Theorem 6 Let $A\in L^{n\times n}$, and A is symmetric, then the maximal solution of $AX=X$ is $(a^{(2)}_{11},a^{(2)}_{22},\ldots,a^{(2)}_{nn})^T$, where $a^{(2)}_{ii}$ denotes a element of A^2 on main diagonal.

Proof For $\forall i.j\in\{1,2,\ldots,n\}$,

$a_{ij}^{(2)}=\overset{n}{\underset{h=1}{V}}(a_{ih}\ a_{hj}) \leqslant \overset{n}{\underset{h=1}{V}} a_{ih}=\overset{n}{\underset{h=1}{V}}(a_{ih}\wedge a_{hi})=a_{ii}^{(2)}$, hence A^2

is diagonally dominant, but the maximal solution of $A^2X=X$ is $(a_{11}^{(2)},a_{22}^{(2)},\ldots,a_{nn}^{(2)})^T$. By [3] proposition III-1, the maximal solution of AX=X same is $(a_{11}^{(2)},a_{22}^{(2)},\ldots,a_{nn}^{(2)})^T$.

Deduction Let $A\in L^{nxn}$, and A is symmetric, then

$E^{(1)}\subset E^{(2)}=E^{(4)}=E^{(6)}=\ldots=E^{(2n)}=\ldots$

$E^{(1)}\subset E^{(3)}=E^{(5)}=E^{(7)}=\ldots=E^{(2n+1)}=\ldots$

Example 1 Let $A=\begin{pmatrix}0.1 & 0.2 & 0.3\\ 0.2 & 0.5 & 0.2\\ 0.3 & 0.2 & 0.4\end{pmatrix}$

to find the maximal solution of AX=X.

Solve A is symmtric, easy know

$$A^2=\begin{pmatrix}0.3 & 0.2 & 0.3\\ 0.2 & 0.5 & 0.2\\ 0.3 & 0.2 & 0.4\end{pmatrix}$$

By theorem 6, the maximal solution of AX=X is $(0.3,0.5,0.4)^T$.

Theorem 7 If $A\in L^{nxn}$, and A is trensitive, then $E^{(1)}=E^{(2)}=\ldots=E^{(n)}=\ldots$

Proof By [3] proposition III-1, can know $E^{(1)}\subset E^{(n)}$. Now suppose $\vec{X}$ is any a solution of $A^nX=X$, hence $A^n\vec{X}=\vec{X}$ (1)

By [2] , A is transitive, then have $A^n=A^{n+1}$, hence

$A^n\vec{X}=A^{n+1}\ \vec{X}=A\vec{X}$ (2)

Since (1) and (2), can obtain $\vec{X}=A^n\vec{X}=A\vec{X}$, hence $E^{(n)}\subset E^{(1)}$, therefore $E^{(1)}=E^{(n)}$. Only pay attention to $A^n=A^{n+1}=A^{n+2}=\ldots$ can know $E^{(n)}=E^{(n+1)}=E^{(n+2)}=\ldots$. Natural number s of for any satisfied $1<s<n$, easy show $E^{(1)}=E^{(s)}$, sum up to have $E^{(1)}=E^{(2)}=\ldots=E^{(n)}=\ldots$.

Deduction If $A\in L^{nxn}$, and A is transitive, then

$E^{(1)}$ is producted a subspaceby column vector group of A^n, and the maximal element of $E^{(1)}$ is

$(\bigvee_{j=1}^{n} a_{1j}^{(n)}, \bigvee_{j=1}^{n} a_{2j}^{(n)}, \ldots, \bigvee_{j=1}^{n} a_{nj}^{(n)})^T$. A^n can be obtained by "square method" to go through finite operation:

$A \to A^2 \to A^4 \to \ldots \to A^{2k} \to \ldots$

Example 2 Let $A=\begin{pmatrix} 0.4 & 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.55 & 0.3 & 0.8 & 0.6 \end{pmatrix}$

to find solution set and the maximal solution of AX=X.

Solve Obviously, A is transitive, and

$$A^2= \begin{pmatrix} 0.4 & 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.55 & 0.3 & 0.6 & 0.6 \end{pmatrix} \quad A^4= \begin{pmatrix} 0.4 & 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.55 & 0.3 & 0.6 & 0.6 \end{pmatrix}$$

hence, the maximal solution of AX=X is $(0.4, 0.5, 0.2, 0.6)^T$, solution set of AX=X is producted by column vector group:

$$(0.4, 0.5, 0.2, 0.55)^T$$
$$(0.3, 0.3, 0.2, 0.3)^T$$
$$(0.4, 0.5, 0.2, 0.6)^T$$

It is easy proved:

Theorem 8 Let $A, B \in L^{n\times n}$, if $\bar{X}$ is common solution of AX=X and BX=X, then $\bar{X}$ must be a solution of $(A \cup B)X=X$.

Theorem 9 Let $A \in L^{n\times n}$, E^* denotes solution set of $A^*X=X$, then $E^{(1)} \subset E^*$, where $A^*=t(A)$.

Proof First pay attention to $E^{(1)} \subset E^{(k)}$ $(k>1)$ and by theorem 8 can know that any a solution of AX=X is all a solution of $(A \cup A^2)X=X$. By use n times theorem 8, can obtain: Any a solution of AX=X is all a solution of $(A \cup A^2 \cup \ldots \cup A^n)X=X$. But $A^*=A \cup A^2 \cup \ldots \cup A^n$, hence $E^{(1)} \subset E^*$.

From theorem 7, have $E^*=E^{*(n)}$, hence E^* can be obtained by $E^{*(n)}$. Therefore all solutions of AX=X can find in solution set of A*X=X.

References

[1] E. Sanchez, "Resslution eigen Fuzzy sets equation", Fuzzy Sets and Systems. 1(1978),69--74.

[2] H.Hashimoto, Convergence of powers a Fuzzy transitive matrix, ibid, 9(1983), 153--160.

[3] Yang Liangcai Fuzzy eigen equation, convergence of power of a Fuzzy matrix and Fuzzy system's stability, FUZZY MATHEMATICS. 4 (1982) 37--48.