# THE EXTENSION ALGEBRA

Wang Hongxu
Dept. of Basis. Liaoyang College of Petrochemistry
Liaoyang. China
Zhang Hongchen
Liaohua Cultivating and Training Centre for Staff and Workers. Liaoyang. China
Dai Hongchai
Dept. of Basis. Liaoyang College of Petrochemistry
Liaoyang. China

## ABSTRACT

In this paper we introduced first the concept of an extension subset, that is

Definition 2.1 The so called " Extension subset $\tilde{A}$ " in the objects set U is indicated to provided a real number

$$K_{\tilde{A}}(u) \in (-\infty, +\infty)$$

for any $u \in U$, by which the relationship of u and $\tilde{A}$ is described. The mapping

$$K_{\tilde{A}} \quad : \quad U \dashrightarrow (-\infty, +\infty)$$
$$u \mapsto K_{\tilde{A}}(u)$$

is called a dependent function of $\tilde{A}$. The an extension subset over U writed E(U).

Some operation on an E(U) are defined as follows:

Definition 3.1

Inclution : $\tilde{A} \subseteq \tilde{B}$ iff $\forall u \in U$, $K_{\tilde{A}}(u) \leq K_{\tilde{B}}(u)$

Equality : $\tilde{A} = \tilde{B}$ iff $\forall u \in U$, $K_{\tilde{A}}(u) = K_{\tilde{B}}(u)$

Intersection: $\tilde{C} = \tilde{A} \cap \tilde{B}$ iff $\forall u \in U$,

$$K_{\tilde{C}}(u) = \min \{K_{\tilde{A}}(u), K_{\tilde{B}}(u)\}$$

Union : $\tilde{D} = \tilde{A} \cup \tilde{B}$ iff $\forall u \in U$,

$$K_{\tilde{D}}(u) = \max \{K_{\tilde{A}}(u), K_{\tilde{B}}(u)\}$$

Complement: $\tilde{S} = \tilde{A}^c$ iff $\forall u \in U$, $K_{\tilde{S}}(u) = -K_{\tilde{A}}(u)$
here $\tilde{A},\tilde{B},\tilde{C},\tilde{D}$, and $\tilde{S} \in E(U)$.

Lastly, we proved those operations are reasonable.

Keyword: Extension algebra.

## I FOREWORD

Since the mathematician of Germany, G. Cantor established the set theory in 19 century, the classical set theory quickly became the basis of morden mathematics. In fact, the classical set theory was not convinent in many respects.

Example 1.1 The size of the workpiece machined by somebody is $50^{+0.01}_{-0.02}$ , if the workpiece machined is examined , it means and/or unqualified, thus , in the set A of all qualified workpieces, any workpiece a if a is within the range of $50^{+0.01}_{-0.02}$ , $a \in A$ or else, $a \overline{\in} A$. This is the method of description for the classical set.

But we note that in the unqualified products, one type of the workpieces is $\phi < 49.98$ these are waste products, and the other type of them is $\phi > 50.01$, they are also unqualified, but it is possible to become qualified after remachining. We call this type of workpiece as remachining products. It is clear that the waste products and remachining ones are different unqualified products in their charactors. According to the view point of classical set. Obviously, it is not appropriate that we devide only the workpiece as a qualified product and/or unqualified one.

We only can describe the definite concept with the classical set. In the objective world, there are too much fuzzy concept, for example, in the language of mankind, middle aged person, bigger number,..., etc. In order to describe the fuzzy concept. American expert of Cybernetics L.A.Zadeh established the fuzzy set theory in 1965, but in the research of many questions about fuzzy subset, there are also some questions, which is open to question.

Example 1.2 Let the ages is a bouned range $U = [0,100]$. L.A.Zadeh has giveen the member function of the fuzzy subset " young " as following

$$\mu_{\underset{\sim}{Y}}(u) = \int_0^{25} 1/u + \int_{25}^{100} [1 + (u - 50)^2 \cdot 5^{-2}]^{-1}/u \ .$$

The Fig 1.1 is the curve of their function.

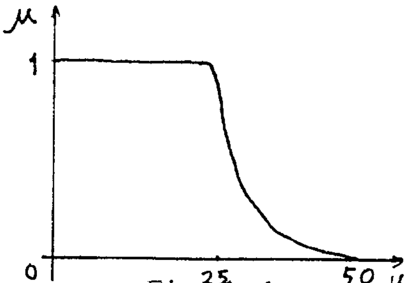


Fig 1.1

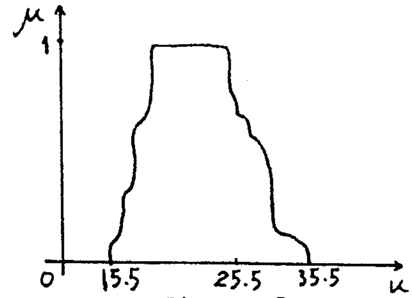


Fig 1.2

Hene, the meaning of " young " at least comparably is of great difference from the Chinese understanding. Chinese classify the ages under 25 years old according to such grades: infancy, teen-ager and young people. Therefor , to define $0 \leq u \leq 25$ as $\mu_{\underset{\sim}{Y}}(u) = 1$ in the same way is not in confirmity with the Chinese custom. Chinese never call a theree-year old child a young man.

Then how to define the " young " to be in confirmity with the Chinese custom ? In the paper [2] Zhang Nanlun finish a fuzzy statistical experiment. In [2] the test group of the Wuhan University statistical curve of member function as Fig 1.2 . We see, in the interval [20, 25.5], $\mu_{\underset{\sim}{A}}(u) = 1$ (here $\underset{\sim}{A}$ as " young " ) as ages above 25.5 years old, $\mu_{\underset{\sim}{A}}(u) > 1$ , and $\mu_{\underset{\sim}{A}}(u) < 1$ when ages under 20 years young.

We put forward one question that when $\mu_{\underset{\sim}{A}}(u) = 0.8$, try to distinguash the age of u, therehence, according to the member function $\mu_{\underset{\sim}{A}}(u)$, we can only draw out the age of u is less than 20 or more than 25.5, we can even

estimate that the age of u is 18 or 20, but actually we can not tell it is 18 or 20 according to $\mu_{\underset{\sim}{A}}(u)$.
In short, to study the fuzzy concepts such as middle age person, tall person, and fat person,...,etc. And so on with the method of fuzzy subset bas got the similar shortcoming.

From the above two examples, we know that it is necessary to establish the new mathematics mode, in order to research the real question.

The Chinese scholar Cai Wen in [3] and [4] first put forward the conception of an extension subset in 1983, this is the better attempt in this respect.

## II THE CONCEPTION OF AN EXTENSION

Definition 2.1 The so called " Extension subset $\tilde{A}$ " in the objects set U is indicated to provided a real number

$$K_{\tilde{A}}(u) \in (-\infty, +\infty)$$

for any $u \in U$, by which the relationship of u and $\tilde{A}$ is described . The mapping

$$K_{\tilde{A}}: \quad U \dashrightarrow (-\infty, +\infty)$$
$$u \mapsto K_{\tilde{A}}(u)$$

is called a dependent function of $\tilde{A}$. The an extension subset over U writed E(U).

An extension subset is described completely by it ralationhip function.

Using the concept of an extension subset, we onec more discussed the preceding examples.

Example 2.1 In example 1.1 we define that

$$K_{\tilde{B}}(u) = \begin{cases} \log_{50.01} u, & 50.01 < u, \\ 1, & 49.98 \leqslant u \leqslant 50.01, \\ \log_{49.98} u, & u < 49.98. \end{cases}$$

We see that $K_{\tilde{B}}(u) = 1$ if $u \in [49.98, 50.01]$. Right now is a qualified product.

$K_{\tilde{B}}(u) > 1$ if $u \in (50.01, +\infty)$. Right now u is a remachining product.

$K_{\tilde{B}}(u) < 1$ if $u \in (-\infty, 49.98]$. Right now u is a waste product.

Using this method we can to describe the question in example 1.1 , and the question is very distinct.

Example 2.2 In the example 1.2 , we define relationship function of " young " as

$$K_{\tilde{A}}(u) = \begin{cases} \log_{25}u , & \text{if } u > 25.5 , \\ 1, & \text{if } 20 \leqslant u \leqslant 25.5 , \\ \log_{20}u , & \text{if } 1 \leqslant u < 20 , \\ 0, & \text{if } 0 \leqslant u < 1 . \end{cases}$$

We see that if $u < 20$ then $K_{\tilde{B}}(u) < 1$. If $u > 25.5$ then $K_{\tilde{B}}(u) > 1$.

We again put forward one question that when $K_{\tilde{A}}(u) = 0.8$, try to distinguash the age of u, by the relationship function of $K_{\tilde{A}}(u)$ we can right away to answer that age of u is 18.

## III THE OPERATIONS AND THE PROPERTIES OF AN EXTENSION SUBSET

Definition 3.1 Let $\tilde{A}, \tilde{B} \in E(U)$ and $\forall u \in U$. We introduce some operations over $E(U)$ as follows:

1° Inclution: $\tilde{A} \subseteq \tilde{B}$ iff $\tilde{A}(u) \leqslant \tilde{B}(u)$ ;

2° Equality: $\tilde{A} = \tilde{B}$ iff $\tilde{A}(u) = \tilde{B}(u)$ ;

3° Intersection : $\tilde{C} = \tilde{A} \cap \tilde{B}$ iff $\tilde{C}(u) = \min\{\tilde{A}(u), B(u)\}$

4° Union : $\tilde{D} = \tilde{A} \cup \tilde{B}$ iff $\tilde{D}(u) = \max\{A(u), B(u)\}$;

5° Complement : $\tilde{S} = \tilde{A}^c$ iff $\tilde{S}(u) = -\tilde{A}(u)$ .

We can prove that:

Theorem 3.1 $(E(U), \subseteq)$ satisfies the following conditions:

(1) $\tilde{A} = \tilde{B}$ iff $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$ ;

(2) $\tilde{A} \subseteq \tilde{A}$ ;

(3) $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{C} \Rightarrow \tilde{A} \subseteq \tilde{C}$ ;

(4) $\tilde{A} \subseteq \tilde{B} \Leftrightarrow \tilde{A} \cup \tilde{B} = \tilde{B} \Leftrightarrow \tilde{A} \cap \tilde{B} = \tilde{A}$ ;

Corollary $(E(U), \subseteq)$ is a partial ordered set .

Theorem 3.2 $(E(U), \cup, \cap, c)$ have following properties:

(E,1) Idempotent laws $\tilde{A} \cap \tilde{A} = \tilde{A}$ , $\tilde{A} \cup \tilde{A} = \tilde{A}$

(E,2) Commutative laws $\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$ , $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$

(E, 3) Associative laws $(\tilde{A} \cup \tilde{B}) \cup \tilde{C} = \tilde{A} \cup (\tilde{B} \cup \tilde{C})$ ;
$(\tilde{A} \cap \tilde{B}) \cap \tilde{C} = \tilde{A} \cap (\tilde{B} \cap \tilde{C})$

(E,4) Absorption laws $\tilde{A} \cup (\tilde{A} \cap \tilde{B}) = \tilde{A}$ , $\tilde{A} \cap (\tilde{A} \cup \tilde{B}) = \tilde{A}$

(E,5) distributive laws $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$;
$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$

(E,6) Involution laws $(\tilde{A}^c)^c = \tilde{A}$

(E, 7) De Morgan laws $(\tilde{A} \cup \tilde{B})^c = \tilde{A}^c \cap \tilde{B}^c$, $(\tilde{A} \cap \tilde{B})^c = \tilde{A}^c \cup \tilde{B}^c$

In $(E(U), \cup, \cap, c)$ generally is not hold following laws. i.e. $\tilde{A} \cup U \neq U$, $\tilde{A} \cap U \neq \tilde{A}$, $\tilde{A} \cup \Phi \neq \tilde{A}$, $\tilde{A} \cap \Phi \neq \Phi$, $\tilde{A} \cup \tilde{A}^c \neq U$ , $\tilde{A} \cap \tilde{A}^c \neq \Phi$.

Definition 3.2 An algebraic system $(L, \cup, \cap, c)$ is called an extension algebra, if it have the properties (E,1) --- (E,7).

Clearly $(E(U), \cup, \cap, c)$ is an extension algebra, and $((-\infty, +\infty), \vee, \wedge, c)$ also is an extension algebra, here define c as: $\forall a \in (-\infty, +\infty)$, $a^c = -a$.

## IV AN EXTENSION SUBSET M(U) WITH A BOUNED RANGE [-M,M] OVER UNIVERSE OF DISCOURSE U

Let $M > 0$ is a positive real number. all extension subset with bouned range $[-M, M]$ over universe of discourse U writed as M(U). i.e.

$$M(U) = \{\tilde{A} \mid -M \leq K_{\tilde{A}}(u) \leq M,\ u \in U,\ \tilde{A} \in E(U)\} .$$

We define that M(U) have same operations with E(U), then $\forall \tilde{A}, \tilde{B} \in M(U)$, $\tilde{A} \cup \tilde{B} \in M(U)$, $\tilde{A} \cap \tilde{B} \in M(U)$, $\tilde{A}^c \in M(U)$, and $M(U) \subseteq E(U)$, and M(U) is proper subset of E(U).

Theorem 4.1 $(M(U), \cup, \cap, c)$ is an extension algebra. And $(M(U), \cup, \cap, c)$ is proper extension subalgebra of $(E(U), \cup, \cap, c)$.

We more introduct following

Definition 4.1 Let $\tilde{A} \in M(U)$, $\tilde{A}$ is called minimum set of

M(U), if $\forall u \in U$, $K_{\tilde{A}}(u) = -M$, writed as $\tilde{A} = \tilde{M}_{\phi}$ .

Let $\tilde{B} \in M(U)$, $\tilde{B}$ is called maximum set , writed as $\tilde{B} = \tilde{M}_U$. if $\forall u \in U$, $K_{\tilde{B}}(u) = M$.

Theorem 4.2 $(M(U), \cap, \cup, c)$ satisfies following laws (M,8)

Zero- one laws: $\tilde{A} \cup \tilde{M}_{\phi} = \tilde{A}$, $\tilde{A} \cap \tilde{M}_{\phi} = \tilde{M}_{\phi}$, $\tilde{A} \cup \tilde{M}_U = \tilde{M}_U$, $\tilde{A} \cap \tilde{M}_U = \tilde{A}$.

Theorem 4.3 $(M(U), \cap, \cup, c)$ is a fuzzy algebra (i.e. a soft algebra).

Proof of these theorems is elementary.

Theorem 4.4 Any an extension subset $\tilde{A} \in M(U)$ is isomorphic to some $\underset{\sim}{A} \in \mathcal{F}(U)$. Here U is universe of discourse U, and $\mathcal{F}(U)$ is set of all fuzzy subsets over U.

Proof. For any $\tilde{A} \in M(U)$, let a mapping

$$f: \quad \tilde{A} \dashrightarrow \underset{\sim}{A},$$

$$K_{\tilde{A}}(u) \mapsto K_{\tilde{A}}(u)/2M + 1/2 = \mu_{\underset{\sim}{A}}(u)$$

then $\underset{\sim}{A} \in \mathcal{F}(U)$. And $\tilde{A}$ is isomorphic to $\underset{\sim}{A}$.

Corollary Fuzzy algebra is isomorphic to proper subalgebra of an extension algebra.

## V THE REASON OF AN OPERATIONS OF AN EXTENSION SUBSET

In this part we should prove that those operations in definition 3.1 are reasonable.

Theorem 5.1 Let both f and g are binary operation, if these satisfise following conditions:

(1) Both f and g are a continuous nondecreasing functions;

(2) Commutative laws: $f(x,y) = f(y,x)$ and $g(x,y) = g(y,x)$;

(3) Both $h(x) = f(x,x)$ and $k(x) = g(x,x)$ are strictly monotone increasing continuous functions;

(4) $f(x,y) \leqslant \min\{x,y\}$ and $g(x,y) \geqslant \max\{x,y\}$;

(5) Both f and g satisfy associative and distributive laws:

$$f(x, f(y,z)) = f(f(x,y),z) ;$$

$g(x,g(y,z)) = g(g(x,y),z)$ ;
$f(x,g(y,z)) = g(f(x,y),\ f(x,z))$ ;
$g(x,f(y,z)) = f(g(x,y),g(x,z))$.

Then both f and g exactly defined respectively by

$$f(x,y) = \min\{x,y\} = x \wedge y$$

and

$$g(x,y) = \max\{x,y\} = x \vee y\ .$$

Proof $1^{0}$ By (1) and (3) we may know that both $h(x)$ and $k(x)$ are on $(-\infty,+\infty)$ strictly continuous increasing functions. Then both $h(x)$ and $k(x)$ are one-to-one mapping from $(-\infty,+\infty)$ to $(-\infty,+\infty)$.

$2^{0}$ Let $h(x) = a$, then

$$h(x)=f(x,x)=a \leq \max\{a,f(a,a)\} \leq g(a,f(a,a))$$
$$=f(g(a,a),g(a,a)) = h(g(a,a))$$

note that $h(x)$ is strictly monotone increasing function, so $x \leq g(a,a)$.

But $g(a,a) = g(h(x),h(x)) = g(f(x,x),f(x,x))$

$$= f(x,g(x,x)) \leq \min\{x,g(x,x) \leq x$$

so $x = g(a,a)$.

Thus $f(x,g(x,x))=g(f(x,x),f(x,x))=g(h(x),h(x))= g(a,a)=x$ .

Use the same method we may prove that $g(x,f(x,x))=x$.

$3^{0}$ $f(a,a)=f(g(a,f(a,a)),a)=f(f(g(a,a),g(a,a)),a)$

$$=f(a,f(g(a,a),g(a,a)))=f(f(a,g(a,a)),g(a,a))$$
$$=f(a,g(a,a))=a$$

Use the same method we may prove that $g(a,a)=a$.

$4^{0}$ $a \leq \max\{a,f(a,b)\} \leq g(a,f(a,b))=f(g(a,a),g(a,b))$

$$= f(a,g(a,b)) \leq \min\{a,g(a,b)\} \leq a$$

then $f(a,g(a,b))=a$.

Similary: $g(a,f(a,b))=a$.

$5^{0}$ We fix " a " from (1) know that $g(a,y)$ is continuous function on $(-\infty,+\infty)$. For $b \geqslant a$ there is $b \geqslant c \geqslant a$ hold

$$g(a,c) = b.$$

From (1), $f(a,y)$ is nondecreasing function, then

$$f(a,b) \geqslant f(a,a) = a.$$

But $g(a,y)$ is also nondecreasing function, then

$$g(a,c) \geqslant g(a,a) = a.$$

Thus $f(a,b)=f(a,g(a,c)) \leq \min\{a,g(a,c)\} = a$.

so $f(a,b) = a=\min\{a,b\}$.

Use the same method we may prove that if $a \geqslant b$, have $f(a,b) = b = \min\{a,b\}$.

Therefore $f(a,b) = \min\{a,b\}$.

Likewise: $g(a,b) = \max\{a,b\}$.

Theorem 5.2 Let $h: (-\infty,+\infty) \rightarrow (-\infty,+\infty)$ satisfies following conditions:

(1) h is continuous monotone increasing function;

(2) $\forall x \in (-\infty,+\infty)$, $h(h(x)) = x$ holds;

(3) $\forall x \in (-\infty,+\infty)$, $h(-x) = -h(x)$ holds,

Then $h(x) = -x$.

Proof For $x_0 \in (-\infty,+\infty)$ hypothesize $h(x_0) = y_0 \neq -x_0$.

(i) Let $y_0 > -x_0$, $x_1 = -x_0$ and $h(x_1) = y_1$, then

$$h(y_1) = h(h(x_1)) = x_1$$

so

$$y_1=h(h(y_1))=h(x_1)=h(-x_0)=-h(x_0)=-y_0 < x_0$$

Thereupon

$$-x_0 < y_0=h(x_0) < h(y_1) = x_1 = -x_0$$

produce contradictory.

(ii) Let $y_0 < -x_0$, use the same method we may prove that produce contradictory, also.

Therefor $h(x_0) = -x_0$.

The aforementioned theorem5.1 and theorem 5.2 show that the operetions in the definition 3.1 are reasonable.

## REFERENCES

[1] Zadeh. L.A., The Concept of A Linguistic Variable and Its Application To Approximate Reasosing, Ame-

rican Elsevier Publishing Company, Inc. 1975

[2] Zhang Nanlun, The Membershis And Probabllity Characterisistics of Randon Appearances, Journal of Wuhan Institute of Building Materials, 1981, (1),(2),(3)

[3] Cai Wen, Introduction of Extension Set, BUSETAL , n° 19 (1984)

[4] Cai Wen, Extensiom set , Fuzzy Set And Classcal Set, FIrst Congress of International Fuzzy Systems Assocication, Spain, 1985