ABSOLUTE CONTINUITY AND EXTENSION OF FUZZY MEASURES

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Since Sugeno [2] introduced the concept of fuzzy measure, the extension problem of non additive measure has been considered and a lot of interisting results on it have been given in some papers. Wang [4] gave an extension theorem for a class of fuzzy measures, called quasi-measure, and therefore, as a special case, solved the extension problem of Sugeno's g_λ-fuzzy measures from a semi-ring onto a σ-ring. Wang [5],[6] and [7] discussed the extensions of possibility measures and consonant belief functions from an arbitrary class of subsets of a nonempty set X onto the power set $\mathcal{P}(X)$. Song [2] investigated the extension problem for a class of fuzzy measures which is more general then that one in [4]. In this paper, we introduce a concept of absolute continuity of nonnegative set functions, and give a necessary and sufficient condition for extending a fuzzy measure from an algebra onto a σ-algebra.

Let X be a nonempty set, let $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{F}$ be nonempty classes of subsets of X, and $\mathcal{A}$ be an algebra, $\mathcal{F}$ be a σ-algebra containing $\mathcal{A}$. We denote $\mathcal{A}_\sigma = \{ B \mid \exists \{A_n\} \subset \mathcal{A}$, such that $A_n \nearrow B \}$.

Definition 1. A set function $\mu : \mathcal{E} \longrightarrow [0, \infty)$ is called fuzzy measure on $\mathcal{E}$, if it satisfies the following conditions:

(1) $\mu(\phi) = 0$, when the empty set $\phi \in \mathcal{E}$;

(2) Monotonicity, i.e., $\forall A \in \mathcal{E}$, $\forall B \in \mathcal{E}$, $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$;

(3) Continuity from below, i.e., $\forall A_n \in \mathcal{E}$, $n=1,2,\cdots$, $\forall A \in \mathcal{E}$,

$$A_n \nearrow A \Longrightarrow \mu(A) = \lim_{n\to\infty} \mu(A_n);$$

(4) Continuity from above, i.e., $\forall\, A_n \in \mathcal{E}$, $n=1,2,\cdots$, $\forall\, A \in \mathcal{E}$,

$$A_n \searrow A \Longrightarrow \mu(A) = \lim_{n\to\infty} \mu(A_n).$$

And a set function satisfying the conditions (1)—(3) is called lower semi-continuous fuzzy measure, or LSC-fuzzy measure for short.

Definition 2. A nondecreasing set function $\mu : \mathcal{E} \longrightarrow [0,\infty)$ is called to be lower (resp. upper) consistent on $\mathcal{E}$, if $\forall\, B \in \mathcal{E}$, $\forall\, A_n \in \mathcal{E}$, $n=1,2,\cdots$,

$$A_n \nearrow \bigcup_{n=1}^{\infty} A_n \supset B \Longrightarrow \lim_{n\to\infty} \mu(A_n) \geqq \mu(B)$$

$$(\text{resp. } A_n \searrow \bigcap_{n=1}^{\infty} A_n \subset B \Longrightarrow \lim_{n\to\infty} \mu(A_n) \leqq \mu(B)\ ).$$

Lemma 1. Let $\mu : \mathcal{E} \rightarrow [0,\infty)$ be a nondecreasing set function. If $\mathcal{E}$ is closed under the formation of finite intersection (resp. finite union), then, for μ on $\mathcal{E}$, the lower (resp. upper) consistency is equavalent to the continuity from below (resp. from above).

Proof. Suppose μ is continuous from below on $\mathcal{E}$. $\forall\, B \in \mathcal{E}$, $\forall\, A_n \in \mathcal{E}$, $n=1,2,\cdots$, if $A_n \nearrow \bigcup_{n=1}^{\infty} A_n \supset B$, then $A_n \cap B \nearrow B$. By using the monotonicity and the continuity from below of μ, we have

$$\lim_{n\to\infty} \mu(A_n) \geqq \lim_{n\to\infty} \mu(A_n \cap B) = \mu(B),$$

that is, μ is lower consistent. The converse implication relation is obvious.

The proof for upper consistency is similar. ∎

Theorem 1. If μ is a LSC-fuzzy measure on $\mathcal{A}$, then μ may be extended to a LSC-fuzzy measure on $\mathcal{A}_\sigma$ uniquely.

Proof. $\forall\, B \in \mathcal{A}_\sigma$, define $\mu^*(B) = \lim_{n\to\infty} \mu(A_n)$ when $A_n \nearrow B$ and

$\{A_n\} \subset \mathcal{A}$. This definition is unambiguous. In fact, if there exist two sequences $\{A_n\}$ and $\{A_n'\}$ in $\mathcal{A}$, such that both $A_n \nearrow B$ and $A_n' \nearrow B$, then, for any positive integer n_0, $A_n \nearrow B \supset A_{n_0}'$, and by using Lemma 1, we have

$$\lim_{n\to\infty} \mu(A_n) \geqq \mu(A_{n_0}'),$$

therefore,

$$\lim_{n\to\infty} \mu(A_n) \geqq \lim_{n\to\infty} \mu(A_n').$$

The converse inequality holds too. Consequently,

$$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_n').$$

We turn to prove the monotonicity of μ^* on $\mathcal{A}_\sigma$ now. Suppose $A \in \mathcal{A}_\sigma$, $B \in \mathcal{A}_\sigma$, and $A \subset B$. Then, there exist $\{A_n\} \subset \mathcal{A}$ and $\{B_n\} \subset \mathcal{A}$, such that $A_n \nearrow A$ and $B_n \nearrow B$. For any positive integer n_0, since $B_n \nearrow B \supset A \supset A_{n_0}$, we have

$$\lim_{n\to\infty} \mu(B_n) \geqq \mu(A_{n_0}),$$

and therefore

$$\mu^*(B) = \lim_{n\to\infty} \mu(B_n) \geqq \lim_{n\to\infty} \mu(A_n) = \mu^*(A).$$

The continuity from below of μ^* may be proved as follows. Suppose $\{A_n \mid n=0,1,2,\cdots\} \subset \mathcal{A}_\sigma$, and $A_n \nearrow A_0$. By the construction of $\mathcal{A}_\sigma$, for every $n=0,1,2,\cdots$, $\exists \{A_{ni} \mid i=1,2,\cdots\} \subset \mathcal{A}$, such that $A_{ni} \nearrow A_n$. By the zig-zag diagonal method, write $B_1 = A_{11}$, $B_2 = A_{12}$, $B_3 = A_{21}$, $B_4 = A_{13}$, $B_5 = A_{22}$, $B_6 = A_{31}$, $B_7 = A_{14}$, $\cdots$, and denote $B_n' = \bigcup_{i=1}^{n} B_i$, then $B_n' \nearrow \bigcup_{n=1}^{\infty} A_n = A_0$, and therefore,

$$\mu^*(A_0) = \lim_{n\to\infty} \mu(B_n').$$

Observing the fact that, for any positive integer n_0, $\exists\, j = j(n_0)$ such that $B_{n_0}' \subset A_j$, we have, by the monotonicity of μ^*,

$$\mu(B_{n_0}') = \mu^*(B_{n_0}') \leqq \mu^*(A_j).$$

Consequently,

$$\mu^*(A_0) \leq \lim_{j\to\infty} \mu^*(A_j).$$

The converse inequality is assured by the monotonicity of μ^*.

μ^* is an extension of μ, because they coincide on $\mathcal{A}$.

The uniqueness of extension is obvious. ▌

Definition 3. Let μ and ν be two fuzzy measures on $\mathcal{E}$. We say that μ is absolutely continuous with respect to ν, in symbols $\mu \ll \nu$, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\mu(F) - \mu(E) < \varepsilon$, whenever $E \in \mathcal{E}$, $F \in \mathcal{E}$, $E \subset F$ and $\nu(F) - \nu(E) < \delta$.

The concept of absolute continuity given in the above definition is a generalization of that one in the classical measure theory (cf. [1]).

Theorem 2. Let μ be a fuzzy measure on $\mathcal{A}$. μ can be extended onto $\mathcal{A}_\sigma$, if and only if there exists a fuzzy measure ν on $\mathcal{A}_\sigma$ such that $\mu \ll \nu$ on $\mathcal{A}$. The extension is unique and it preserves the absolute continuity with respect to ν.

Proof. Since $\mu \ll \mu$ for any fuzzy measure μ on $\mathcal{A}$, the necessity of the condition in this theorem is evident.

For the sufficiency, we need prove only the continuity from above of μ^* given in the proof of Theorem 1. Suppose $\{A_n\} \subset \mathcal{A}_\sigma$ and $A_n \searrow A_0 \in \mathcal{A}_\sigma$. Take set sequences $\{A_{ni} \mid i=1,2,\cdots\} \subset \mathcal{A}$, which satisfy $A_{ni} \nearrow A_n$, for every $n=0,1,2,\cdots$. Since $A_0 \subset A_n$, $\forall n=1,2,\cdots$, we may assume that $A_{0i} \subset A_{ni}$, $\forall i,n=1,2,\cdots$, without any loss of generality. As $\mu \ll \nu$ on $\mathcal{A}$, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $\mu(F) < \mu(E) + \frac{\varepsilon}{2}$ whenever $E \in \mathcal{A}$, $F \in \mathcal{A}$, $E \subset F$ and $\nu(F) < \nu(E) + \delta$. By using the continuity of ν and the definition of μ^* on $\mathcal{A}_\sigma$, there exist N and N', such that

$$\nu(A_N) < \nu(A_0) + \frac{\delta}{2},$$

$$\nu(A_0) < \nu(A_{0N'}) + \frac{\delta}{2}$$

and

$$\mu^*(A_N) < \mu(A_{NN'}) + \frac{\varepsilon}{2} .$$

Thus, we have

$$\nu(A_{NN'}) \leq \nu(A_N) < \nu(A_{0N'}) + \delta,$$

and therefore,

$$\mu(A_{NN'}) < \mu(A_{0N'}) + \frac{\varepsilon}{2} .$$

Consequently,

$$\mu^*(A_N) < \mu(A_{0N'}) + \varepsilon \leq \mu^*(A_0) + \varepsilon .$$

Observing the monotonicity of μ^*, we obtain

$$\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A_0).$$

Using an analogous method, it is easy to prove $\mu^* \ll \nu$ on $\mathcal{A}_\sigma$.

The uniqueness of the extension has been shown in Theorem 1. ∎

To extend a fuzzy measure from an algebra onto a σ-algebra containing it, we need to introduce a new concept of $\mathcal{A}_\sigma$-approachability of a fuzzy measure on a σ-algebra.

Definition 4. A fuzzy measure μ on $\mathcal{F}$ is called to be $\mathcal{A}_\sigma$-approachable, if $\forall A \in \mathcal{F}$, $\forall \varepsilon > 0$, $\exists B \in \mathcal{A}_\sigma$, such that $B \supset A$ and $\mu(B) \leq \mu(A) + \varepsilon$.

As a main result in this paper, we give the following extension theorem.

Theorem 3. A fuzzy measure μ on $\mathcal{A}$ may be extended to an $\mathcal{A}_\sigma$-approachable fuzzy measure on $\mathcal{F}$, if and only if there exists an $\mathcal{A}_\sigma$-approachable fuzzy measure ν on $\mathcal{F}$, such that $\mu \ll \nu$ on $\mathcal{A}$. The extension is unique and it preserves the absolutely continuity with respect to ν .

Proof. The necessity is evident.

For the sufficiency, Theorem 2 tells us that μ may be extended to a fuzzy measure μ^* on $\mathcal{A}_\sigma$ uniquely, and $\mu^* \ll \nu$ on

$\mathcal{A}_\sigma$. If we define, $\forall A \in \mathcal{F}$,

$$\mu^{**}(A) = \inf\{\mu^*(B) \mid A \subset B \in \mathcal{A}_\sigma\},$$

then μ^{**} is nondecreasing, and it coincides with μ^* on $\mathcal{A}_\sigma$. Observing that ν is $\mathcal{A}_\sigma$-approachable, and using an analogous method given in Theorem 2, we can prove that μ^{**} is continuous on $\mathcal{F}$, and therefore, it is a fuzzy measure on $\mathcal{F}$. Obviously, μ^{**} is $\mathcal{A}_\sigma$-approachable, and it is the unique extension with $\mathcal{A}_\sigma$-approachability.

We can also prove the absolutely continuity of μ^{**} with respect to ν in a similar means. ▌

Since a classical measure on $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$, is $\mathcal{A}_\sigma$-approachable (cf. [1]), we have the following corollary.

Corollary 1. A fuzzy measure μ on $\mathcal{A}$ may be extended onto $\sigma(\mathcal{A})$ uniquely, if there exists a finite measure ν on $\mathcal{A}$, such that $\mu \ll \nu$ on $\mathcal{A}$.

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