ON FUZZY PATH CONNECTEDNESS

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1. INTRODUCTION

In [3], Zheng has introduced and studied a concept of fuzzy path connected topological space. In this paper, we aim to introduce and study a new concept of fuzzy path connected topological space and obtain its various characterizations.

2. PRELIMINARIES

All undefined fuzzy topological concepts and notations we make use of, are standard by now. We do not insist that a fuzzy topology must contain all constants, as in Lowen [1]. However, we recall the following from Pu and Liu [2] and Zheng [3].

<u>Definition 2.1</u> [2]. Two fuzzy sets A and B in an fts (X, t) are said to be <u>Q-separated</u> (<u>Separated</u>) iff there exist t-closed (t-open) fuzzy sets U and V in (X, t) such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \emptyset = B \cap U$.

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It is obvious that A and B are Q-separated iff $\bar{A} \cap B = \phi = A \cap \bar{B}$.

<u>Definition 2.2</u> [2,3] An fts (X, t) is called <u>connected</u> (<u>0-connected</u>) iff there do not exist non-empty fuzzy sets A and B in (X, t) such that A and B are Q-separated (separated) in (X, t) and $X = A \cup B$.

It can be easily shown that both A and B in def. 2.2 are crisp subsets of X.

Also, note that if A and B are crisp, then A and B are Q-separated iff A and B are separated (c.f. [2, Prop. 9.2]). Hence we have the following

<u>Theorem 2.1</u> An fts (X, t) is connected iff (X, t) is 0-connected.

For a fuzzy set A in X, we write A_o for the set $A^{-1}(0,1]$ and call it the <u>support</u> of A and for $A \subseteq X$, 1_A will denote the characteristic function of A.

<u>Definition 2.3</u> [3.] Let (X, T) be a topological space. The collection $\bar{T} = \{ A: A \text{ is a fuzzy set in } X \text{ and } A_o \in T \}$ is a fuzzy topology on X called the <u>fuzzy topology on X introduced by T.</u>

<u>Theorem 2.2</u> Let (X, T) be a topological space. Then (i) (X, T) is connected iff $(X, \bar{T})$ is connected.

(ii) (X, T) is connected iff $(X, \bar{T})$ is 0-connected.

The proof is easy and hence is omitted.

Let I = [0, 1] and e be the euclidean subspace topology on I. Then since (I, e) is connected, by theorem 2.2, $(I, \bar{e})$ is connected and 0-connected.

3. FUZZY PATH CONNECTEDNESS

Definition 3.1. Let (X, t) be an fts. A fuzzy path in X is a fuzzy continuous function $f : (I, \bar{e}) \longrightarrow (X, t)$. The crisp singleton f(o) and f(1) are respectively called the initial and terminal points of the fuzzy path f.

Definition 3.2. An fts (X, t) is said to be fuzzy path connected iff for each x, y $\in$ X, there exists a fuzzy path f in X such that f(o) = x and f(1) = y.

Lemma 3.1 Let (X, T) and (Y, S) be topological spaces. Then the following statements are equivalent :

(i) $f : (X, T) \longrightarrow (Y, S)$ is continuous

(ii) $f : (X, \bar{T}) \longrightarrow (Y, \bar{S})$ is fuzzy continuous

(iii) $f : (X, \bar{T}) \longrightarrow (Y, \omega(S))^{+}$ is fuzzy continuous.

Proof

(i) $\Longrightarrow$ (ii) : Let $U \in \bar{S}$. Then $U_o \in S \Rightarrow f^{-1}(U_o) \in T$.

+ $\omega(S) = \{ f:(X,T) \longrightarrow [0,1] : f \text{ is l.s.c.} \}$, See [1].

But $f^{-1}(U_o) = (f^{-1}(U))_o$ whence $f^{-1}(U) \in \overline{T}$ showing that $f : (X, \overline{T}) \longrightarrow (Y, \overline{S})$ is fuzzy continuous.

(ii) $\Longrightarrow$ (iii) : This is obvious since $\omega(S) \subseteq \overline{S}$.

(iii) $\Longrightarrow$ (i) : Let $U \in S$. Then $1_U \in \omega(S) \Longrightarrow f^{-1}(1_U) \in \overline{T} \Longrightarrow (f^{-1}(1_U))_o \in T$. But $(f^{-1}(1_U))_o = f^{-1}(1_U)_o = f^{-1}(U)$. Thus $f^{-1}(U) \in T$ and therefore $f : (X, T) \longrightarrow (Y, S)$ is continuous.

Using lemma 3.1, the following theorem can now easily be proved.

<u>Theorem 3.1</u>

(i) A topological space (X, T) is path connected iff $(X, \omega(T))$ is fuzzy path connected.

(ii) A topological space (X, T) is path connected iff $(X, \overline{T})$ is fuzzy path connected.

<u>Theorem 3.2</u> Continuous image of fuzzy path connected fts is fuzzy path connected.

<u>Proof.</u> Let (X, t) be fuzzy path connected and $f : (X, t) \longrightarrow (Y, t_1)$ be fuzzy continuous and onto. We show that (Y, t_1) is fuzzy path connected. Let $a, b \in Y$. Since f is onto, $f^{-1}(a)$ and $f^{-1}(b)$ are non-empty. Let $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$. Then since (X, t) is fuzzy path connected, there exists $g : (I, \overline{e}) \longrightarrow (X, t)$ such that $g(o) = x$ and $g(1) = y$.

Put h = fg. Then clearly $h : (I, \bar{e}) \longrightarrow (Y, t_1)$ is a fuzzy path such that $h(o) = fg(o) = a$ and $h(1) = fg(1) = b$. Thus (Y, t_1) is fuzzy path connected.

In an fts (X, t), define a relation $\sim$ on X as follows: for $x, y \in X$, $x \sim y$ iff there exists a fuzzy path f in X from x to y (i.e., $f(o) = x$ and $f(1) = y$).

<u>Theorem 3.3.</u> $\sim$ is an equivalence relation.

<u>Proof.</u> Reflexivity follows trivially by considering constant fuzzy paths. For symmetry, let f be a fuzzy path in X from x to y. Define $g : (I, \bar{e}) \longrightarrow (X, t)$ by $g(r) = f(1-r)$ for $r \in I$. Then g is fuzzy continuous and hence a fuzzy path in X from y to x. For transitivity, let f be a fuzzy path from x to y and g be a fuzzy path from y to z. Define $h : (I, \bar{e}) \longrightarrow (X, t)$ by :

$$h(r) = \begin{cases} f(2r), & \text{if } o \le r \le 1/2 \\ g(2r-1), & \text{if } 1/2 \le r \le 1. \end{cases}$$

Then h is fuzzy continuous and hence a fuzzy path in X from x to z (by lemma 1 and 2 of [3]). So $x \sim z$. Hence $\sim$ is an equivalence relation on X.

<u>Theorem 3.4</u> A fuzzy path connected fts. (X, t) is connected(and hence O-connected).

<u>Proof.</u> Let (X, t) be fuzy path connected.

Fix some $x \in X$. For each $y \in X$, there is a fuzzy path in X, say f_y, such that $f_y(o) = x$ and $f_y(1) = y$. Let R_y be the range of f_y. Then R_y is connected since it is the continuous image of $I = [o, 1]$, which is connected. Clearly, $X = \bigcup_{y \in X} R_y$. Since $R_y \cap R_z \neq \emptyset$ for any $y, z \in X$, R_y and R_Z cannot be Q-separated. Hence by [2, theorem 10.2], (X, t) is connected.

In the sequel, by 'subspace' of an fts, we mean 'crisp subspace' only.

The following result follows easily in view of theorem 3.3.

<u>Theorem 3.5.</u> Let $\{A_j : j \in J\}$ be a family of fuzzy path connected subspaces of an fts (X, t) and $\bigcap_j A_j \neq \emptyset$. Then $\bigcup_j A_j$ is fuzzy path connected (as a subspace).

<u>Theorem 3.6</u> A product of fts is fuzzy path connected iff each coordinate fts is fuzzy path connected.

Proof. Let $\{(X_j, t_j) : j \in J\}$ be a family of fts and let $(X, t) = \prod_j (X_j, t_j)$.

If (X, t) is fuzzy path connected then since the projections are fuzzy continuous, (X_j, t_j), for each $j \in J$, is fuzzy path connected. Conversely, let for each $j \in J$, (X_j, t_j) be fuzzy path connected. Let $x, y \in X$ and let $x = (x_j)$ and $y = (y_j)$, $j \in J$. For each $j \in J$, there exists a fuzzy path

f_j in (X_j, t_j) from $p_j(x)$ to $p_j(y)$. Define $f : (I, \tilde{e}) \longrightarrow (X, t)$ by $f(r)(j) = f_j(r)$, for all $j \in J$ and $r \in I$; thus $f(r) = (f_j(r))$. Now we see that $p_j \circ f = f_j$ and that f is fuzzy continuous. Hence f is a fuzzy path in (X, t). Clearly, $f(o) = x$ since $f(o)(j) = f_j(o) = p_j(f(o)) = p_j(x) = x_j$. Similarly, $f(1) = y$. Thus (X, t) is fuzzy path connected.

<u>Definition 3.3</u> Let (X, t) be an fts. Then the maximal fuzzy path connected subspace (D, t_D) of (X, t) is called <u>fuzzy path component</u> of (X, t).

It is clear that the fuzzy path components of an fts (X, t) are precisely the equivalence classes of X under the equivalence relation $\sim$ (of theorem 3.3) on X.

<u>Theorem 3.7</u> Let (X, t) be an fts. Then (i) each fuzzy path connected subspace of (X, t) is contained in some fuzzy path component of (X, t); (ii) If A and B are two fuzzy path components of (X, t), then either $A = B$ or $A \cap B = \emptyset$.

<u>Proof.</u>

(i) Let A be a fuzzy path connected subspace of (X, t) and $M = \{A_j : A_j$ is a fuzzy path connected subspace of (X, t) with $A \subseteq A_j\}$. Then by theorem 3.5, $\cup M$ is fuzzy path connected in (X, t). Now it is clear that $\cup M$ is a fuzzy path component of (X, t) and $A \subseteq \cup M$.

(ii) Let $A \neq B$. If $A \cap B \neq \emptyset$, then $A \cup B$ is fuzzy path connected in (X, t) and either $A \subsetneqq A \cup B$ or $B \subsetneqq A \cup B$, which contradicts the maximality of A or B. Hence $A = B$.

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