FUZZY MAPPINGS - SEQUENCES AND SERIES

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1. Introduction.

The aim of this paper is to explain a concept of definition of the sequences and series of fuzzy mappings. The results which we have received in this paper are similer in forme to the classical theory. In the Section 2 we consider the sequence of fuzzy mappings and its properties. The Section 3 is devoted to series of fuzzy mappings. In this paper we will use the notions and definitions from [1], [2], [3] and [4].

2. Sequences of fuzzy mappings.

A sequence of fuzzy mappings is a function whose domain is the set of positive integers and whose range is a set of fuzzy mappings. We denote a sequence of fuzzy mappings by $\{F_n\}$.

Corresponding to a number t in the domain of each of the terms of the sequence of fuzzy mappings $\{F_n\}$, there is a sequence of fuzzy numbers $\{F_n(t)\}$(see [1]). If $\{F_n(t)\}$ converges for each number t in a set T and we let $F(t) = \lim_{n\to\infty} F_n(t)$, then we say that $\{F_n\}$ converges pointwise to F on T.

We now define another type of convergence for sequences of fuzzy mappings - uniform convergence - and we will show that the limit of a uniformly convergent sequence of continuous with respect to the metric D fuzzy mappings is continuous.

A sequence of fuzzy mappings $\{F_n\}$ convergens uniformly to F on a set T if for each $\varepsilon > 0$ there exists a number N such that for all $t \in T$

$$D(F_n(t),F(t)) < \varepsilon \qquad \text{whenever } n > N.$$

It is clear that if $\{F_n\}$ is uniformly convergent to F on T, then the sequence is pointwise convergent to F on T. But pointwise convergence of $\{F_n\}$ to F on T does not imply uniform convergence of the sequence $\{F_n\}$ on T.

Theorem 2.1. If the sequence $\{F_n\}$ converges uniformly to F on the set T and each of the terms F_n is continuous with respect to the metric D on T then F is continuous on T.

Proof. Let t_0 be any number in T and take a number $\varepsilon > 0$. We wish to show that there is a number $\delta > 0$ such that

$$D(F(t),F(t_0)) < \varepsilon \qquad \text{whenever } t \in T \text{ and } 0 < |t-t_0| < \delta .$$

The uniform convergence of $\{F_n\}$ to F on T implies that for some positive integer n

$$D(F_n(t),F(t)) < \varepsilon/3 \qquad \text{for all } t \in T.$$

Since F_n is continuous at t_0, there exists a $\delta > 0$ such that

$$D(F_n(t),F_n(t_0)) < \varepsilon/3 \qquad \text{whenever } t \in T \text{ and } |t-t_0| < \delta .$$

Therefore, for all $t \in T$ and $|t-t_0| < \delta$

$$\begin{aligned} D(F(t),F(t_0)) &\leq D(F(t),F_n(t)) + D(F_n(t),F(t_0)) \leq \\ &\leq D(F(t),F_n(t)) + D(F_n(t),F_n(t_0)) + \\ &+ D(F_n(t_0),F(t_0)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon . \end{aligned}$$

This completes the proof.

Lemma 2.1. If F and G are integrable fuzzy mappings over $[a,b]$ and for any $t \in [a,b]$

$$D(F(t),G(t)) < \varepsilon/(b-a)$$

then

$$D\left(\int_a^b F, \int_a^b G\right) < \varepsilon .$$

Proof. If for any $t \in [a,b]$, $D(F(t),G(t)) < \varepsilon/(b-a)$ then for any partition P of $[a,b]$

$$D(U_i(F),U_i(G)) < \varepsilon / (b-a)$$

and

$$D(L_i(F),L_i(G)) < \varepsilon / (b-a).$$

Then for any $\alpha \in [0,1]$ we have

$$d(U(F,P)^{\alpha},U(G,P)^{\alpha}) = \max \Big(\Big| \sum_{i=1}^{n} \underline{U_i(F)^{\alpha}}(t_i-t_{i-1}) - \sum_{i=1}^{n} \underline{U_i(G)^{\alpha}}(t_i-t_{i-1}) \Big|,$$

$$\Big| \sum_{i=1}^{n} \overline{U_i(F)^{\alpha}}(t_i-t_{i-1}) - \sum_{i=1}^{n} \overline{U_i(G)^{\alpha}}(t_i-t_{i-1}) \Big| \Big)$$

$$\leq \max \Big(\sum_{i=1}^{n} \big| (\underline{U_i(F)^{\alpha}-U_i(G)^{\alpha}}) (t_i-t_{i-1}) \big|,$$

$$\sum_{i=1}^{n} \big| (\overline{U_i(F)^{\alpha}- U_i(G)^{\alpha}}) \cdot (t_i-t_{i-1}) \big| \Big) <$$

$$< \max \Big(\sum_{i=1}^{n} \frac{\varepsilon}{b-a} (t_i-t_{i-1}), \sum_{i=1}^{n} \frac{\varepsilon}{b-a} (t_i-t_{i-1}) \Big) = \varepsilon .$$

Hence

$$D(U(F,P),U(G,P)) < \varepsilon \quad .$$

The proof that $D(L(F,P),L(G,P)) < \varepsilon$ is similar. From the above inequalities it follows that

$$D\Big(\underline{\int_a^b} F, \underline{\int_a^b} G\Big) < \varepsilon$$

and

$$D\Big(\overline{\int_a^b} F, \overline{\int_a^b} G\Big) < \varepsilon \quad .$$

Hence

$$D\Big(\int_a^b F, \int_a^b G\Big) < \varepsilon \quad .$$

This completes the proof.

Theorem 2.2. If the sequence $\{F_n\}$ of fuzzy mappings converges uniformly to the fuzzy mapping F on the closed interval $[a,b]$ and if each of the terms F_n is integrable over $[a,b]$, then F is integrable over $[a,b]$. Also if

$$G_n(u) = \int_a^u F_n \quad \text{and} \quad G(u) = \int_a^u F, \quad u \in [a,b],$$

then $\{G_n\}$ converges to G on $[a,b]$.

Proof. First we show that F is integrable over $[a,b]$. Take $\varepsilon > 0$. Since $\{F_n\}$ converges uniformly to F on $[a,b]$, for some n

$$D(F_n(t),F(t)) < \varepsilon/3(b-a) \quad \text{for all } t \in [a,b].$$

Thus, for any partition P of $[a,b]$,

$$D(L_i(F),L_i(F_n)) < \varepsilon/3(b-a)$$

and

$$D(U_i(F),U_i(F_n)) < \varepsilon/3(b-a)$$

and, hence

$$D(L(F,P),L(F_n,P)) < \varepsilon/3$$

and

$$D(U(F,P),U(F_n,P)) < \varepsilon/3.$$

Since F_n is integrable over $[a,b]$, $D(U(F_n,P),L(F_n,P)) < \varepsilon/3$ for some partition P of $[a,b]$ (see [3]). For this partition we have

$$D(U(F,P),L(F,P)) < \varepsilon.$$

This shows that F is integrable over $[a,b]$. We now show that $\{G_n\}$ converges uniformly to G on $[a,b]$, that is, for any $\varepsilon > 0$ there exists a number N such that for all $u \in [a,b]$

$$D\left(\int_a^u F, \int_a^u F_n\right) < \varepsilon \qquad \text{whenever } n > N.$$

Since $\{F_n\}$ converges uniformly to F on $[a,b]$, there exists a number N such that for all $t \in [a,b]$

$$D(F_n(t), F(t)) < \varepsilon \qquad \text{whenever } n > N.$$

Therefore, for all $n > N$ and for all $u \in [a,b]$

$$D\left(\int_a^u F_n, \int_a^u F\right) < \varepsilon .$$

This completes the proof.

3 . Series of fuzzy mappings.

If $\{F_k\}$ is a sequence of fuzzy mappings, then the series $\sum_{k=1}^{\infty} F_k$ is the sequence $\{S_n\}$ where $S_n = \sum_{k=1}^{n} F_k$. Notice that now S_n is a fuzzy mapping: it is the fuzzy mapping with domain $\bigcap_{k=1}^{n} \mathcal{D}_{F_k}$ ($\mathcal{D}_{F_k}$ the domain of F_k) and rule of correspondence $S_n(t) = \sum_{k=1}^{n} F_k(t)$. If, for each t in a set T, $\sum_{k=1}^{\infty} F_k(t)$ (that is, $\{S_n(t)\}$) converges to a fuzzy number $F(t)$, then we say that the series converges pointwise to F on T.

The series $\sum_{k=1} F_k$ converges uniformly to F on the set T if for each $\varepsilon > 0$, there exists a number N such that for all $t \in T$

$$D(S_n(t), F(t)) < \varepsilon \qquad \text{whenever } n > N.$$

That is, $\sum_{k=1}^{\infty} F_k$ converges uniformly to F on T if the sequence $\{S_n\}$ converges uniformly to F on T.

There is a test for uniform convergence of a series of fuzzy mappings that is similar to the comparison test for convergence of a series of fuzzy numbers (see [4]).

Theorem 3.1. If $\sum_{k=1}^{\infty} X_k$ converges and if $|F_k(t)| \leq X_k$ for all $t \in T$ and for all k sufficiently large, then $\sum_{k=1}^{\infty} F_k$ converges uniformly on T.

Proof. Let $F(t) = \sum_{k=1}^{\infty} F_k(t)$, $S_n(t) = \sum_{k=1}^{n} F_k(t)$, $X = \sum_{k=1}^{\infty} X_k$ and $S_n = \sum_{k=1}^{n} X_k$. If $|F_k(t)| \leq X_k$ for all $k > N_1$, then for all $t \in T$ and for all positive integers n and m with $m > n > N_1$ we have for all $\alpha \in [0,1]$

$$d(S_m(t)^{\alpha}, S_n(t)^{\alpha}) = d\Big(\sum_{k=1}^{m} F_k(t)^{\alpha}, \sum_{k=1}^{n} F_k(t)^{\alpha}\Big) =$$

$$= \max\Big(\Big|\sum_{k=1}^{m} \underline{F_k(t)}^{\alpha} - \sum_{k=1}^{n} \underline{F_k(t)}^{\alpha}\Big|, \Big|\sum_{k=1}^{m} \overline{F_k(t)}^{\alpha} - \sum_{k=1}^{n} \overline{F_k(t)}^{\alpha}\Big|\Big) \leq$$

$$\leq \max\Big(\sum_{k=n+1}^{m} |\underline{F_k(t)}^{\alpha}|, \sum_{k=n+1}^{m} |\overline{F_k(t)}^{\alpha}|\Big) \leq$$

$$\leq \max\Big(\sum_{k=n+1}^{n} \underline{X}_k^{\alpha}, \sum_{k=n+1}^{n} \overline{X}_k^{\alpha}\Big) = \max\big(\underline{S_m^{\alpha} - S_n^{\alpha}}, \overline{S_m^{\alpha} - S_n^{\alpha}}\big) =$$

$$= d(S_m^{\alpha}, S_n^{\alpha}).$$

Therefore,

$$\lim_{m\to\infty} d(S_m(t)^{\alpha}, S_n(t)^{\alpha}) \leq \lim_{m\to\infty} d(S_m^{\alpha}, S_n^{\alpha}),$$

that is

$$d(F(t)^{\alpha}, S_n(t)^{\alpha}) \leq d(X^{\alpha}, S_n^{\alpha}).$$

Take $\varepsilon > 0$. Since $\lim_{n\to\infty} S_n = X$, there exists a number $N_{\alpha} > N_1$ such that $d(X^{\alpha}, S_n^{\alpha}) < \varepsilon$ whenever $n > N_{\alpha}$. Then, for all $t \in T$

$$d(F(t)^{\alpha}, S_n(t)^{\alpha}) < \varepsilon \quad \text{whenever} \quad n > N_{\alpha}.$$

Hence

$$D(F(t), S_n(t)) < \varepsilon \quad \text{whenever} \quad n > N = \sup_{\alpha\in[0,1]} N_{\alpha}.$$

This shows that $\sum_{k=1}^{\infty} F_k$ is uniformly convergent on T.

We now prove that if a series of continuous fuzzy mappings converges uniformly to a fuzzy mapping F on some set T, then F is continuous on T.

Theorem 3.2. If the series $\sum_{k=1}^{\infty} F_k$ converges uniformly on the set T and each of the terms F_k is continuous on T, then the sum of the series is continuous on T.

Proof. Uniform convergence of the series $\sum_{k=1}^{\infty} F_k$ is equivalent to uniform convergence of the sequence $\{S_n\}$ to F. Since each term F_k of the series is continuous on T, then each term $S_n = \sum_{k=1}^{n} F_k$ of the sequence is continuous on T. Thus, the continuity of F on T follows from Theorem 2.1.

References

1. M.Matłoka, Sequences of fuzzy numbers, BUSEFAL (in print).
2. M.Matłoka, Limits and continuity of the fuzzy functions, BUSEFAL (in print).
3. M.Matłoka, On integral of fuzzy mappings, BUSEFAL (in print).
4. M.Matłoka, On improper integrals of fuzzy mappings and on series of fuzzy numbers, BUSEFAL (in print).