A NOTE ON FIXED POINTS FOR FUZZY MAPPINGS

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ABSTRACT

This paper presents a new fixed point theorem for fuzzy mapping. The result given in this paper improve and perfected a result of Butnariu in [1].

I. INTRODUCTION AND PERLIMINARIES

Fixed point theorems for fuzzy mappings have been considered by Butnariu and the author in [1-3]. The purpose of this paper is to study this problem further. The reslts given in this paper improve and perfected a result of Butnariu in [1].

Let X be a nonempty set. Denote by $\mathcal{F}(X)$ the set of all fuzzy sets on X. If F: $X \to \mathcal{F}(X)$ is a fuzzy mapping, then $F(x)$ (denoted by F_x) is a Fuzzy set on X for each $x \in X$. But $F_x(y)$ denotes subordinate degree of $y \in X$ belongs to fuzzy set F_x. Hence, Fuzzy mapping F on X expressible as a fuzzy set on $X \times X$, i.e. it is defined by the real valued function $F(x,y)=F_x(y)$ from $X \times X$ into $[0,1]$.

Let F: $X \to \mathcal{F}(X)$ be a fuzzy mapping. If $\{y \in X: F_x(y)=\max_{u \in X} F_x(u)\}$ is a nonempty set, then we can define a set-valued mapping $\hat{F}: X \to 2^X$ as follows:

$$\hat{F}(x) = \left\{ y \in X: F_x(y) = \max_{u \in X} F_x(u) \right\},$$

and $\hat{F}$ is called a set-valued mapping induced by the fuzzy mapping F.

$x^* \in X$ is called a fixed point of a fuzzy mapping $F: X \longrightarrow \mathcal{F}(X)$, if

$$F_{x^*}(x^*) \geqslant F_{x^*}(x) \quad \forall\ x \in X.$$

From the definition we can immediately obtain the following result.

Lemma [3]. $x^* \in X$ is a fixed point of a fuzzy mapping F, if and only if x^* is a fixed point of the set-valued mapping $F: X \rightarrow 2^X$, i.e. $x^* \in \hat{F}(x^*)$.

Definition I. Real valued function $f: X \longrightarrow R$ on topological space is called a upper (lower) semi-continuous if

$$\{x \in X: f(x) < r\} \quad (\{x \in X: f(x) > r\})$$

is open set in X for each $r \in R$.

Definition 2 [1]. Let X be a real vector topological space. Fuzzy mapping $F: X \longrightarrow \mathcal{F}(X)$ on X is called closed, if real valued function $F(x,y) = F_x(y)$ is upper semi-continuous on $X \times X$. F is called convex, if for each $x \in X$, $F_x \in \mathcal{F}(X)$ is a convex fuzzy set on X, i.e. for any $y, Z \in X$, $t \in [0,1]$ we have inequalily

$$F_x(ty+(1-t)z) \geqslant \min\{F_x(y), F_x(z)\}.$$

II MAIN RESULTS

Theorem 1 (kakutani-ky Fan) [4]. Let X be a locally convex Hausdorff real vector topological space. C is a nonempty compace convex subset of X. Suppose that $T: C \longrightarrow 2^C$ is a

Kakutani set-valued mapping, i.e. T satisfy the following conditions:

(i) For any $x \in C$, Tx is a nonempty compact convex set.

(ii) The graph of T:

$$G(T) = \{(x,y) : y \in Tx\}$$

is closed in $X \times X$.

Then T has a fixed point x^* in C, i.e., $x^* \in Tx^*$.

The following theorem was proved in [1].

Theorem 2. (see theorem 2.4 in [1]). Let X be a locally convex Hausdorff real vector topological space. C is a nonempty compact convex subset of X. If F is a convex and closed fuzzy mapping on C.

Then T has a fixed point in C.

In [1] there is no explicit mention of the condition

$$\inf \{F(x,y) : (x,y) \in G(\hat{F})\} > 0.$$

However, as the next example shows, this assumption can't be omitted.

EXAMPLE. Let $X = R$ and $C = [0,1]$. Define $F: C \rightarrow \mathcal{F}(C)$ as follows:

$$F(x,y) = \begin{cases} \frac{1}{2}y, & 0 \leq x < 1, \quad 0 \leq y \leq 1, \\ 1-\frac{1}{2}y, & x = 1, \quad 0 \leq y \leq 1. \end{cases}$$

That is, $F_x(y) = \frac{1}{2}y$, for $0 \leq x < 1$ and $F_x(y) = 1-\frac{1}{2}y$ for $x=1$.

From this, it is easy to prove F_x is a convex Fuzzy set on C for any $x \in C$. Hence, F is convex.

Now we prove that F(x,y) is a upper semi-continuous funtion on $C \times C$. By theorem 1, we can prove that for any $r \in [0,1)$

$$\{(x,y) : F(xy) < r\}$$

is open set in $C \times C$.

In fact, we have

$$\{(x,y):F(x,y)<r\}=\begin{cases}\phi, & \text{for } r=0,\\ \{(x,y): x=0, 0<y<2r\} & \\ \{(x,y): 0<x<1,\ 0<y<2r\} & \text{for } 0<r\leq\frac{1}{2},\\ \{(x,y): x=1,\ 2(1-r)<y<1\} & \text{for } \frac{1}{2}<r\leq 1.\end{cases}$$

Obviously, $\{(x,y): F(x,y)<r\}$ is open set for any $r\in[0,1]$. Hence, fuzzy mapping F is closed.

However, from definition of F(x,y) we know that

$$\hat{F}_x=\begin{cases}\{1\}, & 0\leq x<1\\ \{0\}, & x=1.\end{cases}$$

Hence, it is easy to see that $\hat{F}$ does not have a fixed point. So F(x,y) does not have a fixed point.

Observe that $\inf\{F(x,y) : (x,y)\in G(\hat{F})\}>0$ is not true in this case.

Theorem 3 (Fuzzy generalization of Kakutanky Fan theorem) Let X be a locally convex Hausdorff real vector topological space. C is a nonempty compact convex subset of X. Suppose that F is a fuzzy mapping on C satisfy the following conditions:

(i) For each $x\in C$, $\hat{F}_x$ is a nonempty set.

(ii) $\tilde{F}$ is a convex and closed fuzzy mapping on C.

(iii) $\inf\{F(x,y) : (x,y)\in G(\hat{F})\}>0$.

Then T has a fixed point in C.

Where $\tilde{F}(x,y)=\begin{cases}F(x,y), & x\in\hat{F}_x,\\ 0, & x\bar{\in}\hat{F}_x.\end{cases}$

Obviously, we have

$$F(x,y)\geq\tilde{F}(x,y) \text{ and } \hat{F}=\hat{\tilde{F}}.$$

Hence, $\widetilde{F}$ and F have a same fixed point.

Proof. From Lemma we have only to prove the $\widehat{F}$ have a fixed point in C. It is easily verified that $\widehat{F}$ satisfy the conditions of theorem 1.

1. For any $x, y, z \in C$, $t \in [0,1]$, if $y, z \in \widehat{F}_x$, then

$$\widetilde{F}(x,y) = F(x,y), \quad \widetilde{F}(x,z) = F(x,z) .$$

From $\widetilde{F}$ is convex and definition of $\widehat{F}_x$, we obtain

$$\widetilde{F}x(ty + (1-t)Z) \geqslant \min\{\widetilde{F}_x(y), \widetilde{F}_x(Z)\}$$
$$= \min\{F_x(y), F_x(Z)\} = \max_{u \in c} F_x(u).$$

So $ty + (1-t)Z \in \widehat{F}_x$, i.e. $\widehat{F}_x$ is convex.

2. Suppose that $\{y_\alpha : \alpha \in J\}$ is any net in $\widehat{F}_x$ and its limit is y_0. Then for each $y \in C$ and $\alpha \in J$, we have

$$F(x,y_\alpha) \geqslant F(x,y).$$

since $y_\alpha \in \widehat{F}_x$, hence $\widetilde{F}(x,y_\alpha) = F(x,y_\alpha)$. From $\{(x,y_\alpha) : \alpha \in J\}$ converges to (x,y_0) and upper semi-continuity of $\widetilde{F}(x,y)$, we obtain

$$F(x,y_0) \geqslant \widetilde{F}(x,y_0) \geqslant \lim_{\alpha \in J} \text{sum}\, \widetilde{F}(x,y_\alpha)$$

$$= \lim_{\alpha \in J} \text{sum}\, F(x,y_\alpha) \geqslant F(x,y).$$

Hence, $y_0 \in \widehat{F}_x$, i.e. $\widehat{F}_x$ is closed set in C.

From C is compact, we have $\widehat{F}_x \subset C$. This implies that $\widehat{F}_x$ is compact.

3. Let $r = \inf\{F(x,y) : (x,y) \in G(\widehat{F})\} > 0$,

then $G(\widehat{F}) = \{(x,y) : \widetilde{F}(x,y) \geqslant r\}$.

Since $\widetilde{F}(x,y)$ is upper semi-continuous, hence $G(\widehat{F}) = \{(x,y) : \widetilde{F}(x,y) \geqslant r\}$ is closed set in $X \times X$. By theorem

1, $\hat{F}$ has a fixed point in C. This completes the prove of theorem 3.

Remark. Let T: C⟶2^c be set-valued mapping satisfy the all conditions of theorem 1. then we may define a fuzzy mapping F: C⟶$\mathcal{F}(c)$ as follows:

$$F : x \longrightarrow F_x = \chi_{Tx} \quad \forall x \in C.$$

with χ_{Tx} is a charactevistic function of set Tx. It is easily verified that F satisfy the all conditions of theorem 3. Hence, theorem 3 is fuzzy generalization of Kakutaniky Fan theorem.

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