About the fuzzy description of relative position of patterns*

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1. Introduction

From the very beginning of the research in the field of fuzzy sets one of the main areas of application seemed to be pattern recognition and pattern description (see e.g. [1],[2]).

One of the interesting aspects of pattern description (and evaluation) is the featuring of the relative position of components of a more complicated pattern configuration. Recently we were confronted with the problem, how it was possible to describe the relative position of two patterns in the sense "B is to the left from A" with a degree which runs from 0 to 1.

There was found a result of S.Shaheen [3] in this direction. According to this, if C_A and C_B are the abscisses of the centre of patterns A and B, respectively, and min A, max A, min B and max B the lower and upper boundaries of A and B, respectively, in x, the degree for "B is to the left from A" is

$$D(A \leftarrow B) = \frac{C_A - C_B + 1}{\max B - C_B + C_A - \min A} - \frac{\max A - \min B}{\max B - \min B} \qquad (1)$$

Cf. Fig. 1. The degrees for "to the right", "above" and "below" are similar.

Let us analyze this expression by two examples (see Fig. 2). We have three oblongs: A, B and **C**. According to the figure:

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min A = 3, max A = 6, C_A = 4.5
min B = 2, max B = 4, C_B = 3
min C = 0, max C = 2, C_C = 1

So, using (1) we obtain:

$$D(B\leftarrow A) = \frac{3-4.5+1}{6-4.5+3-2} - \frac{4-3}{6-3} = -0.5\dot{3}$$

$$D(C\leftarrow A) = \frac{1-4.5+1}{6-4.5+1-0} - \frac{2-3}{6-3} = -0.6\dot{6}$$

As we see the results are completely wrong. Let us try now to exchange the parts played by A and B in (1).

$$D(A\leftarrow B) = \frac{C_B-C_A+1}{\max A-C_A+C_B-\min B} - \frac{\max B-\min A}{\max A-\min A} \qquad (2)$$

So, using (2) we obtain:

$$D(B\leftarrow A) = \frac{4.5-3+1}{4-3+4.5-3} - \frac{6-2}{4-2} = -1$$

$$D(C\leftarrow A) = \frac{4.5-1+1}{2-1+4.5-3} - \frac{6-0}{2-0} = -1.2$$

It is clear that the results are again completely wrong.

The problem is, of course, why this is so, and how this contradiction to the original assumption can be eliminated. When we examine the two values in the last version, we must see that the reason of their being negative is the second member in (1). One hypothesis is a printing error in this member: instead of max A-min B must be written max A-min A. Exchanging this part, we obtain:

$$D(A\leftarrow B) = \frac{C_B-C_A+1}{\max A-C_A+C_B-\min B} - \frac{\max B-\min B}{\max A-\min A} \qquad (3)$$

Let us apply now (3) for the two examples on Fig. 2.

$D(B \leftarrow A) = -0.5$
$D(C \leftarrow A) = 0.3$

These values are much more reasonable, but $D(B \leftarrow A) < 0$ even here. A motivation for using such a degree can be that B is overlapping with A so it is not to the left from it (entirely).

Of course, there are further possibilities to transform (1) and, maybe, a suitable expression can be obtained, somehow. But we thought to look for a possibly fuzzy formulation, where not only the centre and boundaries but the whole pattern is considered when deciding about relative position. In the next sections, we shall try to investigate this problem using the fuzzy set concept and the interactive fuzzy algebra (see e.g. [4],[5]).

2. Desirable properties of a relative position descriptor

In the next two sections, we try to introduce independently a relative position descriptor which fulfills the criteria of practical applicability. In order to do this, we shall formulate a few criteria with the help of the next concepts:

Definition 1

If given a pattern A, its fuzzy projection A^f is defined by

$$A^f(x) = (A^{max}(x)-A^{min}(x))/max(A^{max}(x)-A^{min}(x)),$$

where $A^{max}(x)$ is the minimal value for which there is no such point $P(x,y) \in A$ that $A^{max}(x) < y$, if there is at least one such $P(x,y)$. Similarly, $A^{min}(x)$ is the maximum for which there is no such $P(x,y) \in A$ that $A^{min}(x) > y$. If there is no $P(x,y) \in A$, both values are undefinied and

$A^f(x) = \emptyset$.

Definition 2

If given a pattern A, its support is the support of $A^f(x)$ i.e.

$$A^s = \{x \mid A^f(x) > 0\}$$

For both definitions, see Fig. 3. Using the above definitions, we formulate our criteria as follows:

C1. If the support of pattern B is to the left from the support of pattern A (i.e. $\max B^s \leq \min A^s$), the position descriptor $D(B \leftarrow A)$ ("B is to the left from A") must have the value 1.

C2. If the support of B is to the right from the support of A ($\min B^s \geq \max A^s$), $D(B \leftarrow A)$ must assume Ø.

C3. If $\max B^s > \min A^s$ and $\min B^s < \max A^s$, $D(B \leftarrow A)$ must be well between 0 and 1.

C4. If $D(C \leftarrow B) = 1$ $D(C \leftarrow A) \geq D(B \leftarrow A)$.

C5. $D(C \leftarrow B)$ should be independent from the relative position to the origo.

On the basis of the above criteria we shall introduce a possible position descriptor.

3. The position descriptor

In order to give a suitable definition for $D(B \leftarrow A)$, we must define first the fuzzy concept of the area "left from A". For this area, those points which are to the left from the support, must have a membership grade equal 1, while those to the right a grade Ø. Inside the support, the grade must decrease monotonously from left to the right.

Definition 3

If given a pattern A and its fuzzy projection $A^f(x)$, the fuzzy area "not left to A", denoted by $\overline{A^{\leftarrow}}$ is defined by

$$\overline{A^{\leftarrow}}(x) = \int_{-\infty}^{x} A^f(y)dy/(\int_{-\infty}^{+\infty} A^f(y)dy) \tag{4}$$

Definition 4

The fuzzy area $A^{\leftarrow}(x)$ "left to A" is the inverse of $\overline{A^{\leftarrow}}(x)$:

$$A^{\leftarrow}(x) = 1-\int_{-\infty}^{x} A^f(y)dy/(\int_{-\infty}^{\infty} A^f(y)dy) =$$

$$= \int_{x}^{+\infty} A^f(y)dy/(\int_{-\infty}^{+\infty} A^f(y)dy) \tag{5}$$

For the last two definitions see Fig. 4.

So we have the grades over each x in which degree a pattern B is to be found there ($B^f(x)$) and in which degree this place is to the left from A ($A^{\leftarrow}(x)$). Now, we formulate the statement: "x is in the fuzzy projection of B and to the left of A". This can be easily expressed with the help of interactive fuzzy operations [e.g.4]:

$$(B\leftarrow A)^f(x) = B^f(x) \wedge A^{\leftarrow}(x) = B^f(x)\cdot \int_{x}^{+\infty} A^f(y)dy/(\int_{-\infty}^{+\infty} A^f(y)dy) \tag{6}$$

See Fig. 5.

We have come now to the point when a fuzzy object is achieved which is featuring the statement $B\leftarrow A$ in function of x. From this, it is easy to construct the single degree that is looked for. Object B is completely to the left from A if B^s is in the area where $A^{\leftarrow}(x)=1$. Then $B^f(x)=(B\leftarrow A)^f(x)$. Otherwise there are x-s where $(B\leftarrow A)^f(x) < B^f(x)$. Typical is the difference between these two membership functions: it is large, if there is overlapping between A^s and B^s. $(B\leftarrow A)^f(x)$ is identically zero if $B^f(x)$ is in the area where $\overleftarrow{A}(x)$ is $\emptyset$. Considering these properties we have the definition of the position descriptor.

Definition 5

The degree of "B being to the left from A" is described by the ratio of the areals below $(B\leftarrow A)^f(x)$ and $B^f(x)$.

$$D(B\leftarrow A) = \int_{-\infty}^{+\infty}(B\leftarrow A)^f(x)dx/(\int_{-\infty}^{+\infty}B^f(y)dy) =$$
$$= \int_{-\infty}^{+\infty}B^f(x)\cdot\int_{x}^{+\infty}A^f(y)dy/(\int_{-\infty}^{+\infty}A^f(y)dy\cdot\int_{-\infty}^{+\infty}B^f(y)dy) \qquad (7)$$

We shall examine now, whether the properties in criterions C1-C5 are fulfilled.

Criterion 1

If $\max B^s \leq \min A^s$

$$\overleftarrow{A}(x) = \int_{x}^{\infty}A^f(y)dy/\ (\int_{-\infty}^{+\infty}A^f(y)dy) =$$
$$= (\int_{x}^{\infty}A^f(y)dy + \int_{-\infty}^{x}A^f(y)dy)/(\int_{-\infty}^{+\infty}A^f(y)dy = 1,$$

as $\int_{-\infty}^{x}A^f(y)dy = \int_{-\infty}^{x}0dy = 0$

so $(B\leftarrow A)^f(x) = B^f(x)\cdot 1 = B^f(x)$

Than also
$$D(B\leftarrow A) = \int_{-\infty}^{+\infty}B^f(x)dx/(\int_{-\infty}^{+\infty}B^f(x)dx) = 1$$

Criterion 2

If $\min B^s \geq \max A^s$, $\overleftarrow{A}(x) = \int_{x}^{+\infty}A^f(y)dy/(\int_{-\infty}^{+\infty}A^f(y)dy) =$
$= \int_{x}^{+\infty}0dy/c = 0$ for each $x\in B^s$. Than

$(B \leftarrow A)^f(x) = B^f(x) \cdot 0 = 0,$

and so

$$D(B \leftarrow A) = \int_{-\infty}^{+\infty} 0dx / (\int_{-\infty}^{+\infty} B^f(x)dx) = 0$$

Criterion 3

From the conditions in the criterion it follows that $A^S \cap B^S$ is not empty, even that the length of $A^S \cap B^S$ is greater than 0. (Because A and B are connected, also A^S and B^S are connected i.e. intervals, but max $B^S >$ min A^S and so either max $B^S \leq$ max A^S, then max $B^S \in$ (min A^S, max A^S] or max $B^S >$ max A^S but even then min $B^S <$ max A^S i.e. either min $B^S \geq$ min A^S and so min $B^S \in$ [min A^S, max A^S) or min $B^S <$ min A^S, but than $A^S \subset B^S$.)

Let us denote now $A^S \cap B^S$ by S, and S $\setminus$ max S $\setminus$ min S by S'. Then, over S' $\overleftarrow{A}(x) < 1$ and $B^f(x) \geq 0$, so $(B \leftarrow A)^f(x) \leq B^f(x)$. But as $(B \leftarrow A)^f(x) \leq B^f(x)$ everywhere, $\int_{-\infty}^{+\infty} (B \leftarrow A)^f(x)dx < \int_{-\infty}^{+\infty} B^f(x)dx$, and so $D(B \leftarrow A) < 1$.

On the other hand, as min $B^S <$ max A^S, at least over (min B^S, max A^S) both $B^f(x) > 0$, and $\overleftarrow{A}(x) > 0$, so $(B \leftarrow A)^f(x) > 0$, from which

$$\int_{-\infty}^{+\infty} (B \leftarrow A)^f(x)dx > 0$$

and $D(B \leftarrow A) > 0$.

Criterion 4

As $D(C \leftarrow B) = 1$, because of criteria 2 and 3,

max $C^S <$ min B^S.

But so $K_C = \min_{C^S} \overleftarrow{A}(x) \geq \max_{B^S} \overleftarrow{A}(x) = K_B$.

Further on,

$(C \leftarrow A)^f(x) \geq C^f(x) \cdot K_C$

and $(B \leftarrow A)^f(x) \leq B^f(x) \cdot K_B$

$$D(C \leftarrow A) \geq \int_{-\infty}^{+\infty} C^f(x) \cdot K_C dx / (\int_{-\infty}^{+\infty} C^f(x) dx = K_C \geq K_B =$$

$$= \int_{-\infty}^{+\infty} B^f(x) \cdot K_B dx / (\int_{-\infty}^{+\infty} B^f(x) dx) = D(B \leftarrow A)$$

Criterion 5

In (7), nothing depends on the relative position to the origo: The ordinate plays absolutely no part in the definition. The integrals are invariants in respect of any shifting of the curves parallely with the x axis.

So we have proven that all five criteria are fulfilled by Definition 5. We have dealt with only the relative position "to the left". But similarly "to the right" can be defined.

Definition 5A

$$D(A \rightarrow B) = \int_{-\infty}^{+\infty} B^f(x) \cdot \int_{-\infty}^{x} A^f(y) dy / (\int_{-\infty}^{+\infty} A^f(y) dy \cdot \int_{-\infty}^{+\infty} B^f(x) dx)$$

Properties 1-4

It is easy to prove that

$D(A \leftarrow B) = 1 \Rightarrow D(A \rightarrow B) = 0$

and

$D(A \leftarrow B) = 0 \Rightarrow D(a \rightarrow B) = 1.$

Finally,

$D(A \leftarrow B) > 0 \Rightarrow S(A \rightarrow B) < 1$

and

$D(A \leftarrow B) < 1 \Rightarrow D(A \rightarrow B) > 0$

We mention that very similar definitions are possible for "above" and "below" by exchanging the role of x and y.

4. An example

At the end, we want to show a simple example for the introduced descriptor. On Fig.6 we can see two simple patterns consisting of oblongs. They are partly overlapping, but B's bulk is definitely to the left from A. For D(B←A) we get $2.8/3 \cong 0.93$. On Fig.7 the curves belonging to the opposite descriptor D(A→B) are to be seen. For D(B→A) we obtain $\sim 2.33/2.5 = 0.93$. Our results point to an interesting hypothesis:

Property 5

D(A←B) = D(A→B).

Let us prove this very obvious statement.

$$\int_{-\infty}^{+\infty}(B^f(x)\int_{x}^{\infty}A^f(y)dy)dx/(\int_{-\infty}^{+\infty}A^f(y)dy\cdot\int_{-\infty}^{+\infty}B^f(x)dx =$$

$$=\int_{-\infty}^{+\infty}(A^f(y)\cdot\int_{-\infty}^{y}B^f(x)dx)dy/(\int_{-\infty}^{+\infty}B^f(x)dx\cdot\int_{-\infty}^{+\infty}A^f(y)dy)$$

$$\int_{-\infty}^{+\infty}(B^f(x)\cdot\int_{x}^{+\infty}A^f(y)dy)dx = \int_{-\infty}^{+\infty}(A^f(y)\cdot\int_{-\infty}^{y}B^f(x)dx)dy$$

$$\int_{-\infty}^{+\infty}(\int_{x}^{+\infty}B^f(x)\cdot A^f(y)dy)dx = \int_{-\infty}^{+\infty}(\int_{-\infty}^{y}A^f(y)\cdot B^f(x)dx)dy$$

As the patterns are finite (A_1=min A^s), A_2=max A^s, B^1=min B^s), B^2=max B^s) it is equivalently

$$\int_{B_1}^{B_2}(\int_{x}^{A_2} B^f(x)\cdot A^f(y)dy)dx = \int_{A_1}^{A_2}(\int_{B_1}^{y} B^f(x)\cdot A^f(y)dx)dy$$

Which is true, because of Fubini's law.

5. Conclusion

In this paper we have given a new definition for a descriptor of the relative position of arbitrary patterns. First, we established some necessary properties then we proved that all these properties are fulfilled by our definition. Finally, some further interesting features of this descriptor were dealt with. We expect our definition combined with the interactive definitions of fuzzy operations being useful for the application in more complicated pattern evaluation methods.

References

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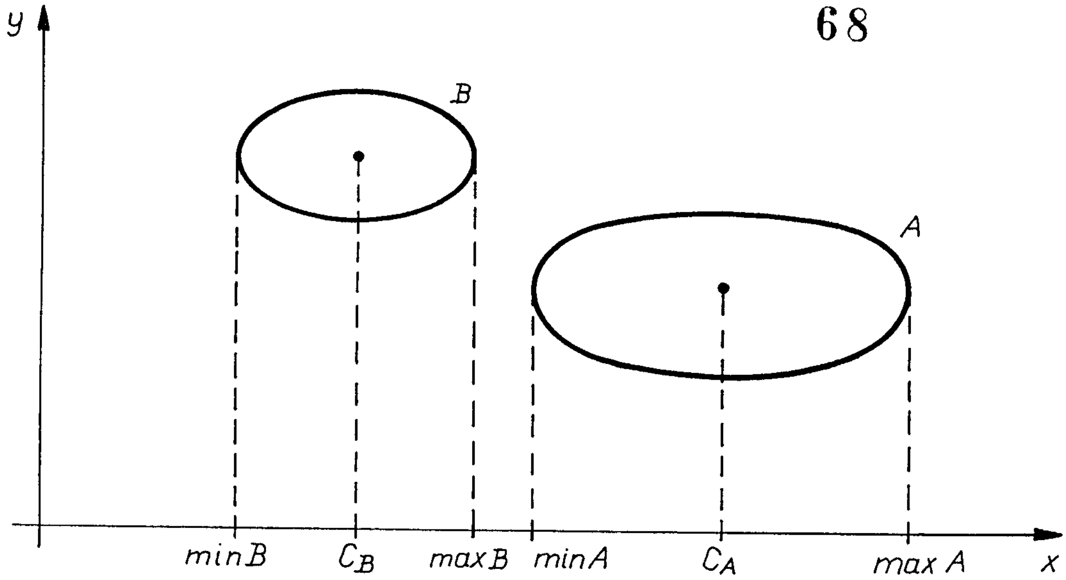

Figure 1.

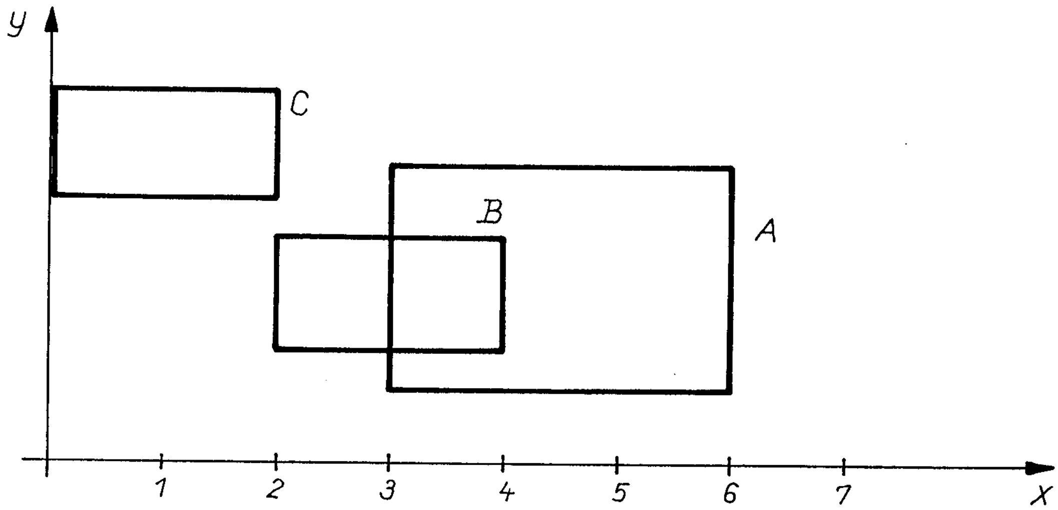

Figure 2.

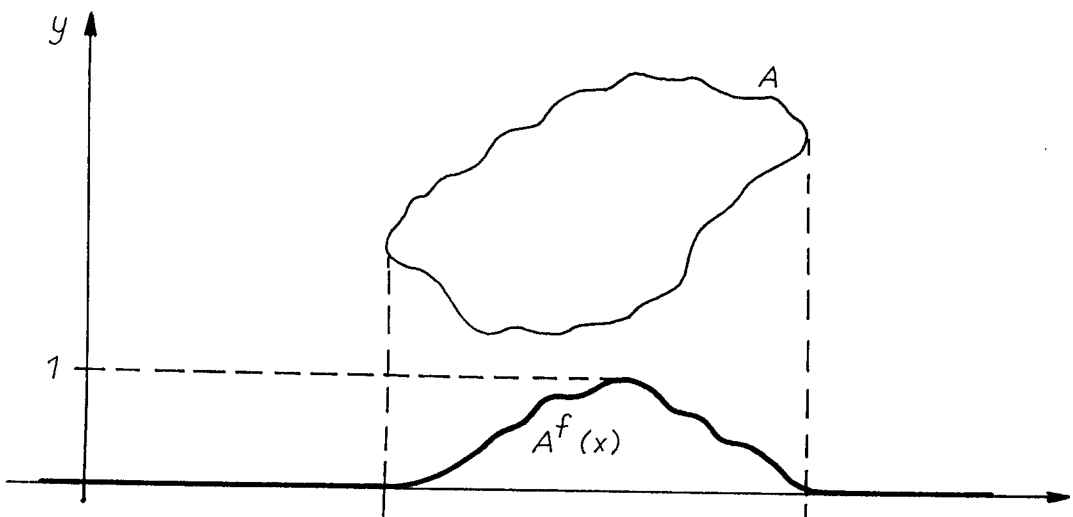

Figure 3.

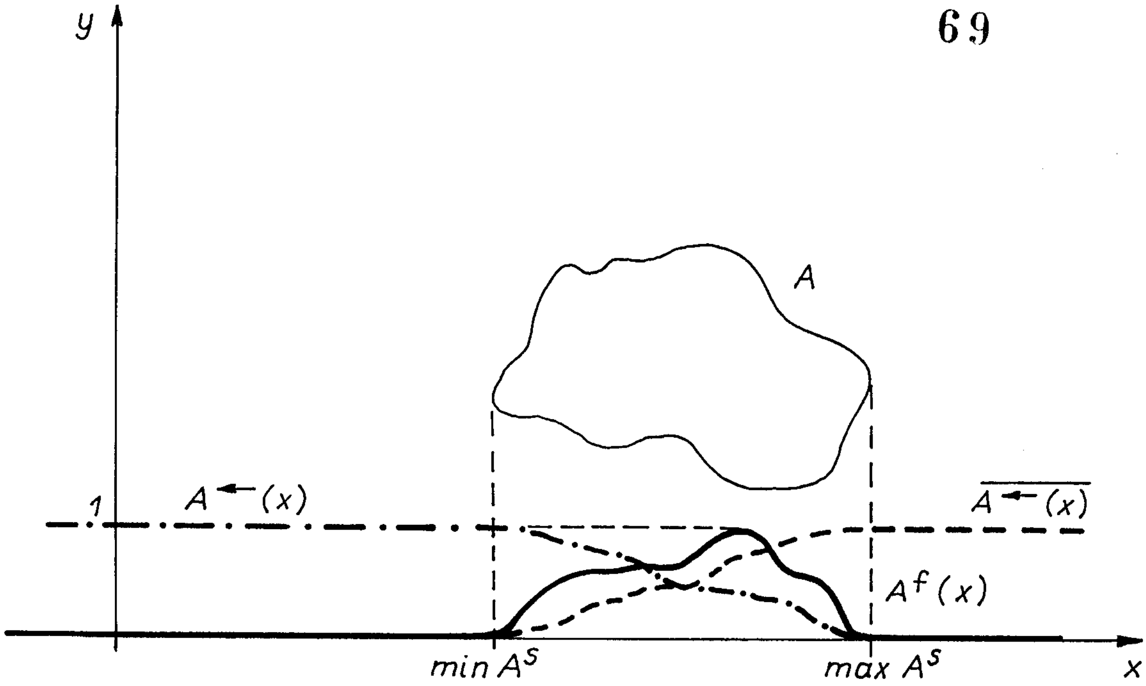

Figure 4.

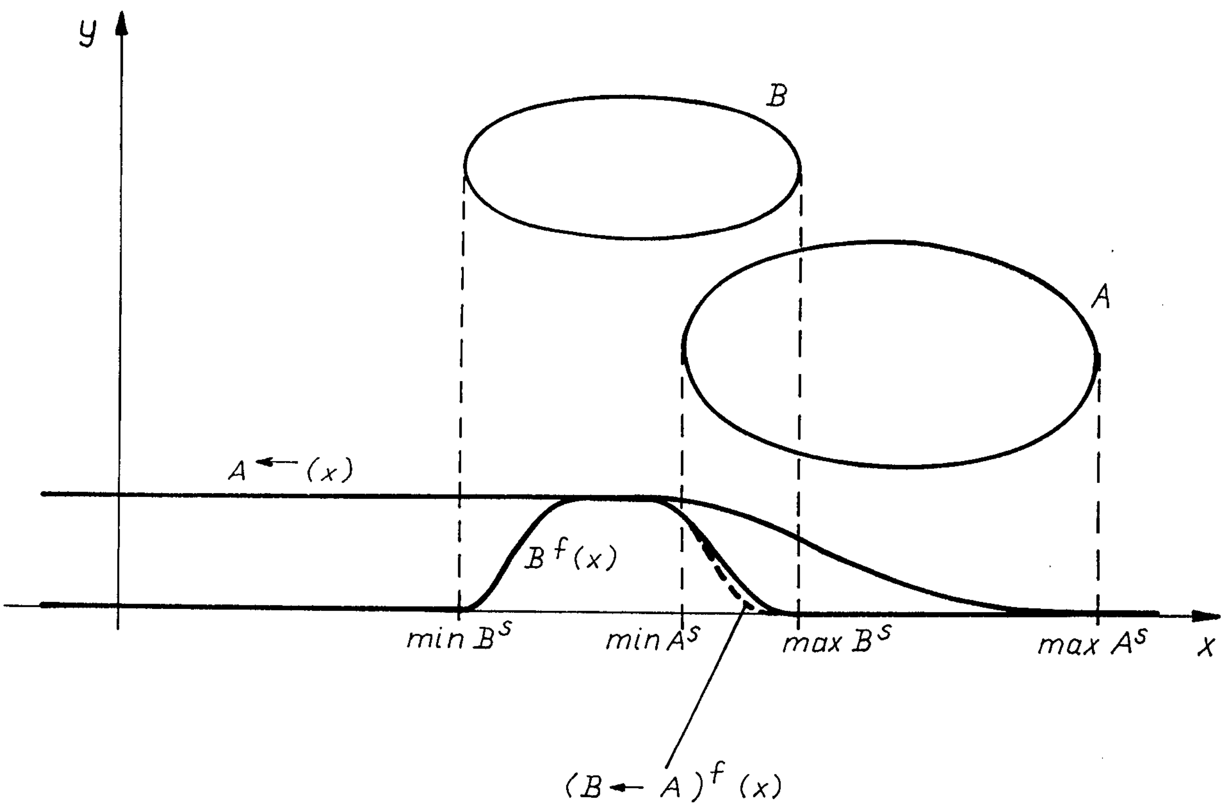

Figure 5.

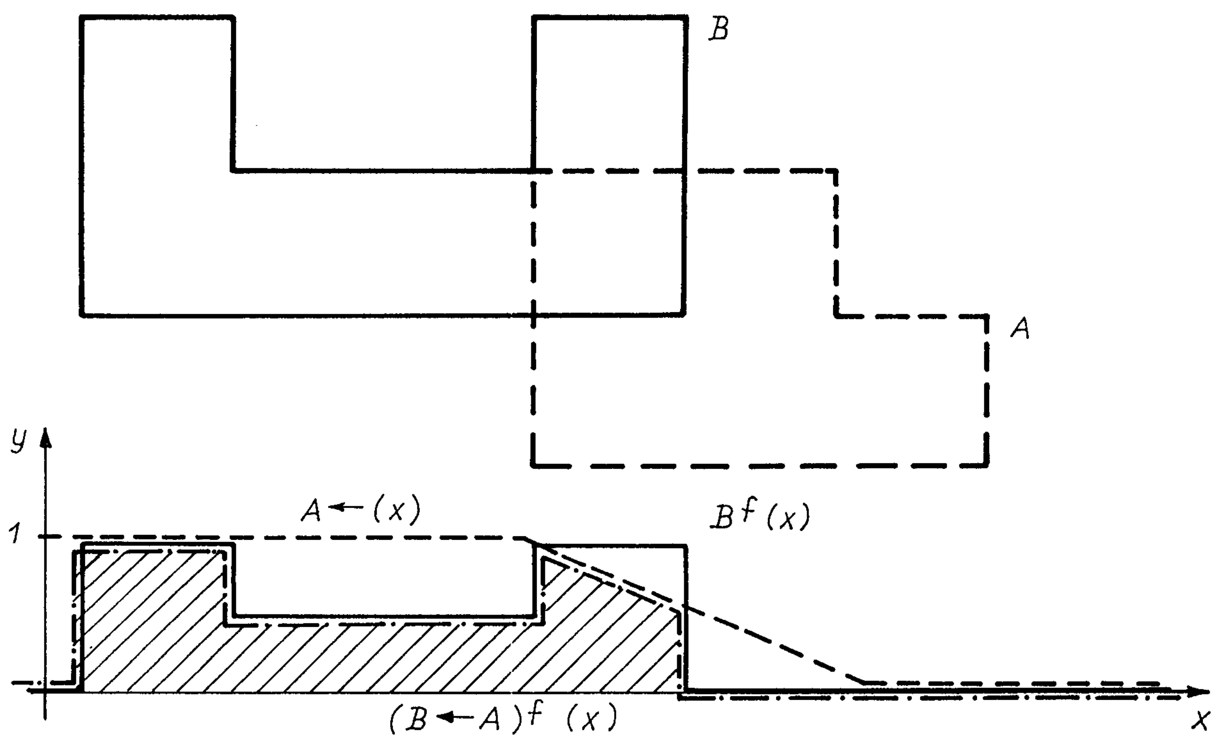

Figure 6.

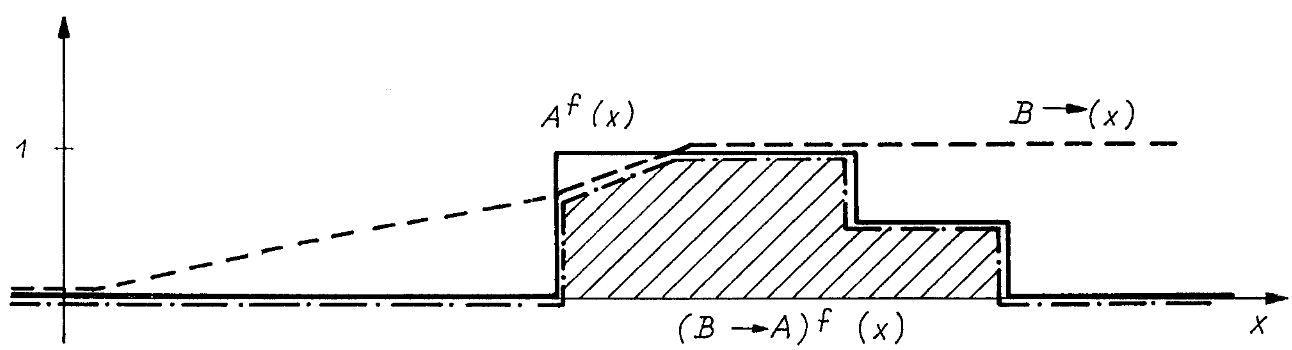

Figure 7.