THE INTERDEPENDENCE BETWEEN ORTHOGONAL PARTITIONS WITH FUZZY EVENTS

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ABSTRACT: In this paper an index quantifying the interdependence between orthogonal partitions with fuzzy events is defined and its properties are investigated.

Keywords: fuzzy event, orthogonal partition, index of the interdependence

RESUME: Dans cet article nous définons un index qu´évalue l´interdépendance parmis deux partitions orthogonaux avec évenements flous, et nous analysons ses proprietées.

Mots clés: évenement flou, partition orthogonaux, index de l'interdepéndance.

1.INTRODUCTION

Let (X, β_X, P) be a probability space, where X is a set in the real line R , β_X is the smallest Borel σ-field on X ,and P a probability distribution on (X, β_X).

A fuzzy event on X is a fuzzy set $\mathcal{X}$ on X whose membership function $\mu_{\mathcal{X}}$ - which associates with each element $x \in X$ a real number in the unit interval $[0,1]$ with the value $\mu_{\mathcal{X}}(x)$ representing the "grade of membership" of x in $\mathcal{X}$ - is Borel-measurable. If $\sum_{\mathcal{X} \in \mathcal{X}^*} \mu_{\mathcal{X}}(x)=1 \quad \forall x \in X$ the family $\mathcal{X}^* = \{\mathcal{X}\}$ is called orthogonal partition with fuzzy events associated with X.

Following Zadeh (6) the probability P associated with the measurable space (X, β_X) induces a probability distribution on $\mathcal{X}^*$ given by

$$\mathcal{P}(\mathcal{X}) = \int_X \mu_{\mathcal{X}}(x)dP(x)$$

and the uncertainty of X^* (Ken Kuriyama(4)) is defined by

$$H(X^*) = -\sum_{X \in X^*} P(X) \log P(X)$$

If (X, β_X, P^1) and (Y, β_Y, P^2) are two probability spaces, the the <u>product probability space</u> is the probability space $(X\times Y, \beta_{X\times Y}, P)$ such that β_X is the σ-field over X induced from $\beta_{X\times Y}$ by the projection of index i=1, β_Y is the σ-field over Y induced from $\beta_{X\times Y}$ by the projection of index i=2 , P^1 is the probability measure on β_X induced from P and P^2 is the probability measure on β_Y induced from P.

Let X^* and Y^* be two orthogonal partitions with fuzzy events associated with X and Y , respectively. We define the <u>combined orthogonal partition</u> $X^* \times Y^*$ as the orthogonal partition on $X\times Y$ formed by all algebraic products of a fuzzy event $X \in X^*$ and a fuzzy event $Y \in Y^*$, denoted by (X, Y) .

Then a probability P associated with $(X\times Y, \beta_{X\times Y})$ induces the probability distribution on $X^* \times Y^*$ given by

$$P(X,Y) = \int_{X\times Y} \mu_X(x) \, \mu_Y(y) \, dP(x,y)$$

Let X^* and Y^* be two orthogonal partitions with fuzzy events associated with X and Y respectively. X^* and Y^* are said to be <u>independent</u> if and only if, $\forall (X,Y) \in X^* \times Y^*,\ P(X,Y) = P(X).P(Y)$

(That is, according to Zadeh´s notion of probabilistic independence,(6),X^* and Y^* are said to be independent if and only if , wathever the fuzzy events $X \in X^*$and $Y \in Y^*$may be, X and Y are independent).

In particular and according to preceding definition, if the probability spaces (X, β_X, P^1) and (Y, β_Y, P^2) are independent then X^* and Y^* are also independent.

Let $X^* \times Y^*$ be a combined orthogonal partition with fuzzy events on $X\times Y$. Then the uncertainty of $X^* \times Y^*$ is given by

$$H(X^* \times Y^*) = \sum_{X \in X^*} \sum_{Y \in Y^*} P(X,Y).\log P(X,Y)$$

Finally, the uncertainty of the orthogonal partition Y^* given the partition X^* is given by

$$H(Y^*/X^*) = \sum_{X \in X^*} P(X) \sum_{Y \in Y^*} P(Y|X).\log P(Y/X)$$

In this framework, an index of the interdependence between the orthogonal partitions with fuzzy events, X^* , Y^* is given by

$$i(X^*,Y^*) = \begin{cases} 1 - \dfrac{H(X^*/Y^*)}{H(X^*)} & \text{if} \quad H(X^*) \geq H(Y^*) \\ 1 - \dfrac{H(Y^*/X^*)}{H(Y^*)} & \text{if} \quad H(X^*) \leq H(Y^*) \end{cases}$$

2.PROPERTIES

In this section, we stablish the properties of the index of interdependence.

Theorem 1

Let X^* and Y^* be two orthogonal partitions with fuzzy events. Then

$$0 \leq i(X^*,Y^*) \leq 1$$

Proof.

Assume $H(X^*) \geq H(Y^*)$; following Okuda, Tanaka and Asai (5) we have

$$H(X^*/Y^*) \leq H(X^*)$$

whence

$$\frac{H(X^*/Y^*)}{H(X^*)} \leq 1$$

consequently

$$1 - \frac{H(X^*/Y^*)}{H(X^*)} \geq 0$$

Therefore

$$i(X^*,Y^*) \geq 0$$

In addition we obtain

$H(X^*) \geq 0$ and $H(X^*/Y^*) \geq 0$

and hence

$$\frac{H(X^*/Y^*)}{H(X^*)} \geq 0$$

Then

$$i(X^*,Y^*) \leq 1$$

On the other hand if $H(X^*) \leq H(Y^*)$, it can be verified with similar arguments that

$$0 \leq i(X^*,Y^*) \leq 1$$

Theorem 2

Let X^* and Y^* be two orthogonal partition with fuzzy events associated with X and Y respectively. Then $i(X^*,Y^*)=0$ if and only if X^* and Y^* are independent orthogonal partitions.

Proof.

Assume that $H(X^*) \geq H(Y^*)$. In this case, the condition $i(X^*,Y^*)=0$ is equivalent to $H(X^*/Y^*)=H(X^*)$

We are now going to verify that $H(X^*/Y^*) = H(X^*)$ if and only if X^* and Y^* are independent. Indeed, $\sum_{X \in X^*} P(X/Y) = \sum_{X \in X^*} P(X)=1$

If we apply Gibbs´Lemma we have,

$$- \sum_{X \in X^*} P(X/Y).\log P(X/Y) \quad - \sum_{X \in X^*} P(X/Y).\log P(X) \qquad (1)$$

with equality if and only if $P(X/Y)=P(X)$

Multiplying the inequality (1) by $P(Y)$ and summing over Y^* we obtain

$$- \sum_{X \in X^*} \sum_{Y \in Y^*} P(Y).P(X/Y)\log P(X/Y) \quad - \sum_{X \in X^*} \sum_{Y \in Y^*} P(Y).P(X/Y).\log P(X)$$

that is,

$$\sum_{X \in X^*} \sum_{Y \in Y^*} P(X,Y).\log P(X/Y) \quad - \sum_{X \in X^*} \sum_{Y \in Y^*} P(X,Y).\log P(X) = - \sum_{X \in X^*} P(X)\log P(X)$$

Therefore $H(X^*/Y^*) \leq H(X^*)$ with equality if and only if $P(X/Y)=P(X)$ i.e. if and only if X^* and Y^* are independent orthogonal partitions.

When $H(Y^*) \geq H(X^*)$ an argument identical to the preceding one may be used to prove the corresponding result.

<u>Theorem 3</u>

Let X^* and Y^* be two orthogonal partitions with fuzzy events. Then $i(X^*,Y^*)=1$ if and only if there exists an invertible mapping g such that $Y^* = g(X^*)$.

Proof.

If there exists an invertible mapping g such that $Y^* = g(X^*)$ it follows that for all $Y \in Y^*$ there exists a fuzzy event X^i in X^* and essentially one, such that

$$P(Y/X^i)=1 \quad \text{and} \quad P(Y/X^j) = 0 \;\; \forall \;\; X^j \neq X^i, \; X^j \in X^*$$

Consequentlly,

$$H(Y^*/X^*) = 0$$

Therefore

$$i(X^*,Y^*) = 1$$

On the other hand, if $i(X^*,Y^*) = 1$ it is easy to prove that there exists an invertible mapping g such that $Y^* = g(X^*)$

Obviously the index $i(X^*,Y^*) = 1$ is a symmetric function of its arguments.

Theorem 4

Let X^* , Y^* and Z^* be three orthogonal partitions with fuzzy events associated with X , Y and Z respectively. Then ,
$d(X^*,Y^*) + d(Y^*,Z^*) \geq d(X^*,Z^*)$ (2) where

$$d(X^*,Y^*) = \begin{cases} H(X^*/Y^*)/H(X^*) & \text{if } H(X^*) \geq H(Y^*) \\ H(Y^*/X^*)/H(Y^*) & \text{if } H(X^*) \leq H(Y^*) \end{cases}$$

(that is, $i(X^*,Y^*) = 1 - d(X^*,Y^*)$)

Proof.

Assume $H(X^*) \geq H(Z^*)$. In order to prove the inequality (2) we have to verify it for each of the three cases:

a) $H(X^*) \geq H(Z^*) \geq H(Y^*)$
b) $H(X^*) \geq H(Y^*) \geq H(Z^*)$
c) $H(Y^*) \geq H(X^*) \geq H(Z^*)$

Case a)

$$d(X^*,Y^*) + d(Y^*,Z^*) = \frac{H(X^*/Y^*)}{H(X^*)} + \frac{H(Z^*/Y^*)}{H(Z^*)}$$

As $H(Y^*) + H(Z^*/Y^*) = H(Z^*) + H(Y^*/Z^*)$ and

$H(Y^*) \leq H(Z^*)$

we have

$H(Z^*/Y^*) \geq H(Y^*/Z^*)$

and hence

$$d(X^*,Y^*) + d(Y^*,Z^*) \geq \frac{H(X^*/Y^*)}{H(X^*)} + \frac{H(Y^*/Z^*)}{H(Z^*)} \geq \frac{H(X^*/Y^*)}{H(X^*)} + \frac{H(Y^*/Z^*)}{H(X^*)}$$

Furthermore

$$H(X^*/Y^*) + H(Y^*/Z^*) \geq H(X^*/Y^*,Z^*) + H(Y^*/Z^*) = H(X^*,Y^*/Z^*) \geq H(X^*/Z^*)$$

Therefore

$$d(X^*,Y^*) + d(Y^*,Z^*) \geq \frac{H(X^*/Z^*)}{H(X^*)} = d(X^*,Z^*)$$

Case b)

Following a similar argument, we have

$$d(X^*,Y^*) + d(Y^*,Z^*) = \frac{H(X^*/Z^*)}{H(X^*)} + \frac{H(Y^*/Z^*)}{H(Y^*)} \geq \frac{H(X^*/Y^*)}{H(X^*)} + \frac{H(Y^*/Z^*)}{H(X^*)} \geq$$

$$\geq \frac{H(X^*/Z^*)}{H(X^*)} = d(X^*,Z^*)$$

Case c)

As in the previous cases, we can immediately verify that

$$d(X^*,Y^*) + d(Y^*,Z^*) = \frac{H(Y^*/X^*)}{H(Y^*)} + \frac{H(Y^*/Z^*)}{H(Y^*)} = \frac{H(Y^*/X^*) + H(Y^*/Z^*)}{H(Y^*)} \geq$$

$$\geq \frac{H(X^*/Z^*)+H(Y^*)-H(X^*)}{H(Y^*)} = 1- \frac{H(X^*)}{H(Y^*)} \left(1- \frac{H(X^*/Z^*)}{H(X^*)}\right) \geq 1-\left(1- \frac{H(X^*/Z^*)}{H(X^*)}\right) =$$

$$= \frac{H(X^*/Z^*)}{H(X^*)} = d(X^*,Z^*)$$

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