Riesz's Theorem and Lebesgue's Theorem on the Fuzzy Measure Space

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Abstract

In this paper, the (pseudo-)autocontinuity and some other concepts of the fuzzy measure are introduced. On the fuzzy measure space $(X, \mathfrak{F}, \underset{\sim}{\mu})$, the concepts of "almost" and "pseudo-almost" are given, and Riesz's theorem and Lebesgue's theorem are proved.

*1. The (pseudo-) autocontinuity of the fuzzy measure

Wang[1] studied some structural characteristics of a fuzzy measure on a classical σ-algebra. In this paper, we shall discuss the fuzzy measure and the convergence of sequence of fuzzy measurable functions on the fuzzy σ-algebra of fuzzy sets.

Throughout this paper, let X be a classical nonempty set.

Definition 1.1. Let $\mathfrak{F}(X)=\{\underset{\sim}{A};\ \underset{\sim}{A}: X\rightarrow[0,1]\}$ be a set of all fuzzy subsets of X, $\mathfrak{F} \subset \mathfrak{F}(X)$. We call $\mathfrak{F}$ a fuzzy σ-algebra, if the properties (1)-(3) are satisfied:

(1) $\phi, X \in \mathfrak{F}$, where $\phi(x)=0$, $X(x)=1$, for every $x \in X$;

(2) If $A \in \mathfrak{F}$, then $A^c \in \mathfrak{F}$;

(3) If $\{A_n\} \subset \mathfrak{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{F}$.

If $A_n \subset A_{n+1}$ (or $A_n \supset A_{n+1}$), $n=1,2,\cdots$, we define $(\lim_{n\to\infty} A_n)(x) = \lim_{n\to\infty} A_n(x)$, for every $x \in X$.

Definition 1.2. Let $\mathfrak{F}$ be a fuzzy σ-algebra, a fuzzy set function $\mu : \mathfrak{F} \to [0,\infty]$ is called a fuzzy measure, if the following conditions are satisfied:

(1) $\mu(\phi)=0$;

(2) whenever $A, B \in \mathfrak{F}$, $A \subset B$, then $\mu(A) \leq \mu(B)$;

(3) Whenever $\{A_n\} \subset \mathfrak{F}$, $A_n \subset A_{n+1}$, $n=1,2,\cdots$, then

$\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

(4) Whenever $\{A_n\} \subset \mathfrak{F}$, $A_n \supset A_{n+1}$, $n=1,2,\cdots$, and there exists n_0, such that $\mu(A_{n_0}) < \infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

If $\mathfrak{F}$ is a fuzzy σ-algebra, and μ is a fuzzy measure, we call $(X, \mathfrak{F}, \mu)$ a fuzzy measure space.

Definition 1.3. The fuzzy measure μ is called null-subtractive, if we have $\mu(A \cap B^c) = \mu(A)$, whenever $A, B \in \mathfrak{F}$, $\mu(B)=0$.

Definition 1.4. A fuzzy measure $\underset{\sim}{\mu}$ is called autocontinuous from above (resp. autocontinuous from below) if we have $\underset{\sim}{\mu}(\underset{\sim}{A} \cup \underset{\sim}{B}_n) \to \underset{\sim}{\mu}(\underset{\sim}{A})$ (resp. $\underset{\sim}{\mu}(\underset{\sim}{A} \cap \underset{\sim}{B}_n^c) \to \underset{\sim}{\mu}(\underset{\sim}{A})$) whenever $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, $\{\underset{\sim}{B}_n\} \subset \underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{\mu}(\underset{\sim}{B}_n) \to 0$. $\underset{\sim}{\mu}$ is called autocontinuous, if it is both autocontinuous from above and autocontinuous from below.

Definition 1.5. Let $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{\mu}(\underset{\sim}{A}) < \infty$. $\underset{\sim}{\mu}$ is called pseudo-autocontinuous from above with respect to $\underset{\sim}{A}$ (resp. pseudo-autocontinuous from below with respect to $\underset{\sim}{A}$) if for any $\{\underset{\sim}{B}_n\} \subset \underset{\sim}{\mathfrak{F}}$

$$\underset{\sim}{\mu}(\underset{\sim}{B}_n \cap \underset{\sim}{A}) \to \underset{\sim}{\mu}(\underset{\sim}{A}) \Longrightarrow \underset{\sim}{\mu}[(\underset{\sim}{A} \cap \underset{\sim}{B}_n^c) \cup \underset{\sim}{E}] \to \underset{\sim}{\mu}(\underset{\sim}{E})$$
$$(\text{resp. } \underset{\sim}{\mu}(\underset{\sim}{B}_n \cap \underset{\sim}{E}) \to \underset{\sim}{\mu}(\underset{\sim}{E}) \;)$$

whenever $\underset{\sim}{E} \in \underset{\sim}{A} \cap \underset{\sim}{\mathfrak{F}} = \{\underset{\sim}{A} \cap \underset{\sim}{D};\ \underset{\sim}{D} \in \underset{\sim}{\mathfrak{F}}\}$.

*2. "Almost" and "pseudo-almost"

Since the fuzzy measures lose the additivity in general, it is necessary to introduce two different concepts, "almost" and "pseudo-almost" on $(X, \underset{\sim}{\mathfrak{F}}, \underset{\sim}{\mu})$.

In the following, let χ_D be the characteristic function of the classical set $D \subset X$.

Definition 2.1. The mapping $f: X \to (-\infty, \infty)$ is called a fuzzy measurable function, if $\chi_{F_\alpha} \in \underset{\sim}{\mathfrak{F}}$, where $F_\alpha = \{x;\ f(x) \geqslant \alpha\}$.

The set of all fuzzy measurable functions is denoted by $\underset{\sim}{M}$.

Definition 2.2. Let $\{f_n\} \subset \underset{\sim}{M}$, $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, $P(x)$ be a proposition with respect to $x \in X$, $D = \{x;\ x \in X,\ P(x) \text{ is true}\}$

(1) If $\underset{\sim}{A} \subset \chi_D$, then we say $P(x)$ is everywhere true on $\underset{\sim}{A}$;

(2) If there exists $\underset{\sim}{E} \in \underset{\sim}{\mathfrak{F}}$ with $\underset{\sim}{\mu}(\underset{\sim}{E})=0$, such that $P(x)$ is everywhere true on $\underset{\sim}{A} \wedge \underset{\sim}{E}^c$, then we say $P(x)$ is almost everywhere true on $\underset{\sim}{A}$;

(3) If there exists $\underset{\sim}{E} \in \underset{\sim}{\mathfrak{F}}$ with $\underset{\sim}{\mu}(\underset{\sim}{A} \wedge \underset{\sim}{E}^c) = \underset{\sim}{\mu}(\underset{\sim}{A})$, such that $P(x)$ is everywhere true on $\underset{\sim}{A} \wedge \underset{\sim}{E}^c$, then we say $P(x)$ is pseudo-almost everywhere true on $\underset{\sim}{A}$.

Particularly, if $P(x)$ is the proposition "$\{f_n\}$ converges to f", we will obtain the concepts of $\{f_n\}$ converge to f "everywhere","almost everywhere","pseudo-almost everywhere" on $\underset{\sim}{A}$. We denote them by "$f_n \xrightarrow{e.} f$ on $\underset{\sim}{A}$" , " $f_n \xrightarrow{a.e.} f$ on $\underset{\sim}{A}$","$f_n \xrightarrow{p.a.e.} f$ on $\underset{\sim}{A}$" respectively.

Definition 2.3. Let $\{f_n\} \subset \underset{\sim}{M}$, $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$.

(1) If $\underset{\sim}{\mu}(\underset{\sim}{A} \wedge \chi_{\{|f_n - f| \geq \varepsilon\}}) \to 0$ for any given $\varepsilon > 0$, then we say $\{f_n\}$ converges in fuzzy measure $\underset{\sim}{\mu}$ to f on $\underset{\sim}{A}$, and denote it by $f_n \xrightarrow{\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$;

(2) If $\underset{\sim}{\mu}(\underset{\sim}{A} \wedge \chi_{\{|f_n - f| < \varepsilon\}}) \to \underset{\sim}{\mu}(\underset{\sim}{A})$ for any given $\varepsilon > 0$, then we say $\{f_n\}$ converges pseudo-in fuzzy measure $\underset{\sim}{\mu}$ to f on $\underset{\sim}{A}$, and denote it by $f_n \xrightarrow{p.\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$.

Proposition 2.4. If $P(x)$ is almost everywhere true on $\underset{\sim}{A}$, $\underset{\sim}{\mu}$ is null- subtractive, then $P(x)$ is pseudo-almost everywhere true on $\underset{\sim}{A}$.

*3. Riesz's theorem on the fuzzy measure space $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu})$

By means of the (pseudo-) autocontinuity of a fuzzy measure and the concepts of "almost" and "pseudo-almost", we may give three forms of generalization for the classical Riesz's theorem.

Lemma 3.1. Let $\{\underset{\sim}{E}_n\} \subset \underset{\sim}{\mathcal{F}}$, $\underset{\sim}{A} \in \underset{\sim}{\mathcal{F}}$. If $\underset{\sim}{\mu}(\underset{\sim}{E}_n) \to 0$, and $\underset{\sim}{\mu}$ is autocontinuous from above (resp. autocontinuous from below), then there exists some subsequence $\{\underset{\sim}{E}_{n_i}\}$ of $\{\underset{\sim}{E}_n\}$, such that $\underset{\sim}{\mu}(\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} \underset{\sim}{E}_{n_i}) = 0$ (resp. $\underset{\sim}{\mu}[\underset{\sim}{A} \cap (\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} \underset{\sim}{E}_{n_i})^c] = \underset{\sim}{\mu}(\underset{\sim}{A})$).

Proof. We only give the proof of $\underset{\sim}{\mu}$ is autocontinuous from below, the other is similar.

Let $\underset{\sim}{\mu}$ be autocontinuous from below, $\underset{\sim}{\mu}(\underset{\sim}{E}_n) \to 0$. For arbitrarily given $\varepsilon > 0$, there exists n_1, such that $\underset{\sim}{\mu}(\underset{\sim}{A} \cap \underset{\sim}{E}_{n_1}^c) > \underset{\sim}{\mu}(\underset{\sim}{A}) - \frac{\varepsilon}{2}$. And for $\underset{\sim}{A} \cap \underset{\sim}{E}_{n_1}^c$, there exists $n_2 > n_1$, such that

$$\underset{\sim}{\mu}[\underset{\sim}{A} \cap (\underset{\sim}{E}_{n_1} \cup \underset{\sim}{E}_{n_2})^c] > \underset{\sim}{\mu}(\underset{\sim}{A} \cap \underset{\sim}{E}_{n_1}^c) - \frac{\varepsilon}{4} > \underset{\sim}{\mu}(\underset{\sim}{A}) - \frac{3}{4}\varepsilon, \cdots$$

and so on. Finally, we obtain a sequence $\{\underset{\sim}{E}_{n_i}\}$, such that $\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcup_{i=1}^{\infty}\underset{\sim}{E}_{n_i})^c]\geqslant\underset{\sim}{\mu}(\underset{\sim}{A})-\varepsilon$. Furthermore, we take a subsequence $\{\underset{\sim}{E}_{n_i}^{(1)}\}$ of $\{\underset{\sim}{E}_n\}$, such that

$\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcup_{i=1}^{\infty}\underset{\sim}{E}_{n_i}^{(1)})^c]\geqslant\underset{\sim}{\mu}(\underset{\sim}{A})-1$. And as $\underset{\sim}{\mu}(\underset{\sim}{E}_{n_i}^{(1)})\to 0$ too, there exists a subsequence $\{\underset{\sim}{E}_{n_i}^{(2)}\}$ of $\{\underset{\sim}{E}_{n_i}^{(1)}\}$, such that $\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcup_{i=1}^{\infty}\underset{\sim}{E}_{n_i}^{(2)})^c]\geqslant\underset{\sim}{\mu}(\underset{\sim}{A})-\frac{1}{2}$. In general, there exists a subsequence $\{\underset{\sim}{E}_{n_i}^{(j)}\}$ of $\{\underset{\sim}{E}_{n_i}^{(j-1)}\}$, such that

$\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcup_{i=1}^{\infty}\underset{\sim}{E}_{n_i}^{(j)})^c]\geqslant\underset{\sim}{\mu}(\underset{\sim}{A})-\frac{1}{j}$, $j=1,2,\cdots$. If we take $n_i=n_i^{(i)}$, then $\{\underset{\sim}{E}_{n_i}\}$ is a subsequence of $\{\underset{\sim}{E}_n\}$, and $\bigcup_{j}^{\infty}\underset{\sim}{E}_{n_i}\subset\bigcup_{i=1}^{\infty}\underset{\sim}{E}_{n_i}^{(j)}$, $j=1,2,\cdots$, consequently,

$\underset{\sim}{\mu}(\underset{\sim}{A})\geqslant\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}\underset{\sim}{E}_{n_i})^c]\geqslant\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcup_{i=1}^{\infty}\underset{\sim}{E}_{n_i}^{(j)})^c]\geqslant\underset{\sim}{\mu}(\underset{\sim}{A})-\frac{1}{j}$, for all $j=1,2,\cdots$. And therefore $\underset{\sim}{\mu}[\underset{\sim}{A}\cap(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}\underset{\sim}{E}_{n_i})^c]=\underset{\sim}{\mu}(\underset{\sim}{A})$.

Lemma 3.2, Let $\{\underset{\sim}{E}_n\}\subset\underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$. If $\underset{\sim}{E}_n\subset\underset{\sim}{A}$, $n=1,2,\cdots$, $\underset{\sim}{\mu}(\underset{\sim}{E}_n)\to\underset{\sim}{\mu}(\underset{\sim}{A})<\infty$, and $\underset{\sim}{\mu}$ is pseudo-autocontinuous from below with respect to $\underset{\sim}{A}$, then there exists some subsequence $\{\underset{\sim}{E}_{n_i}\}$ of $\{\underset{\sim}{E}_n\}$, such that $\underset{\sim}{\mu}(\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}\underset{\sim}{E}_{n_i})=\underset{\sim}{\mu}(\underset{\sim}{A})$.

The proof of Lemma 3.2 is similar to Lemma 3.1.

The following theorem is a generalization of the classical Riesz's theorem.

Theorem 3.3. Let $\{f_n\} \subset M$, $f \in M$, $A \in \mathcal{F}$.

(1) If μ is autocontinuous from above, $f_n \xrightarrow{\mu} f$ on X, then there exists some subsequence $\{f_{n_i}\}$ of $\{f_n\}$, such that $f_{n_i} \xrightarrow{a.e.} f$ on A, whenever $A \in \mathcal{F}$;

(2) If μ is autocontinuous from below, $f_n \xrightarrow{\mu} f$ on X, then there exists some subsequence $\{f_{n_i}\}$ of $\{f_n\}$, such that $f_{n_i} \xrightarrow{p.a.e.} f$ on A, whenever $A \in \mathcal{F}$;

(3) If μ is pseudo-autocontinuous from below with respect to A, $\mu(A) < \infty$, $f_n \xrightarrow{p.\mu} f$ on A, then there exists some subsequence $\{f_{n_i}\}$ of $\{f_n\}$, such that $f_{n_i} \xrightarrow{p.a.e.} f$ on A.

Proof. Let μ be autocontinuous from above, and $f_n \xrightarrow{\mu} f$ on X, then for every $k=1,2,\cdots$, $\mu(\chi_{\{|f_n - f| \geq \frac{1}{k}\}}) \to 0$. There exists n_k respectively, such that $\mu(\chi_{\{|f_{n_k} - f| \geq \frac{1}{k}\}}) < \frac{1}{k}$.

Without any loss of generality, we suppose $n_{k+1} > n_k$, $k=1,2,\cdots$. If we denote $E_k = \chi_{\{|f_{n_k} - f| \geq \frac{1}{k}\}}$, then $\lim_{k \to \infty} \mu(E_k) = 0$. By using Lemma 3.1 there exists a subsequence $\{E_{k_i}\}$ of $\{E_k\}$, such that $\mu(\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_{k_i}) = 0$. For every $A \in \mathcal{F}$, we have $A \cap (\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_{k_i})^c \subset \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} E_{k_i}^c$. Furthermore we prove

$\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E^c_{k_i} \subset \chi_D$, where $D=\{f_{n_{k_i}} \to f\}$. In fact, for any $x \in \bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}\{|f_{n_{k_i}} - f|<\frac{1}{k_i}\}$, there exists $j(x)$, such that $x \in \bigcap_{i=j(x)}^{\infty}\{|f_{n_{k_i}} - f|<\frac{1}{k_i}\}$, namely, $|f_{n_{k_i}}(x) - f(x)|< \frac{1}{k_i}$, as $i \geqslant j(x)$. Thus, for arbitrarily given $\varepsilon > 0$, if we take i_0 such that $\frac{1}{k_{i_0}}<\varepsilon$, then $|f_{n_{k_i}}(x) - f(x)|<\frac{1}{k_i}\leqslant\frac{1}{k_{i_0}}<\varepsilon$, as $i \geqslant j(x)\vee i_0$, namely $x \in D$. Consequently $\underset{\sim}{A}\cap(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}\underset{\sim}{E}_{k_i})^c \subset \chi_D$, that is to say $f_{n_{k_i}}\xrightarrow{a.e.} f$ on $\underset{\sim}{A}$.

(2) and (3) may be proved by using Lemma 2.6, and Lemma 2.7.

§4. Lebesgue's theorem on $(X, \underset{\sim}{\mathfrak{F}}, \underset{\sim}{\mu})$

Now we give two forms of the generalization of the classical Lebesgue's theorem.

Theorem 4.1. Let $\{f_n\}\subset \underset{\sim}{M}$, $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$.

(1) If $f_n\xrightarrow{p.a.e.} f$ on $\underset{\sim}{A}$, then $f_n\xrightarrow{p.\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$;

(2) Let $\underset{\sim}{\mu}$ be null-subtractive, if $f_n\xrightarrow{a.e.} f$ on $\underset{\sim}{A}$, then $f_n\xrightarrow{p.\underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$.

Proof. By using proposition 2.4, we can obtain (2) from (1). Now we prove (1). As $f_n\xrightarrow{p.a.e.} f$ on $\underset{\sim}{A}$, there exists $\underset{\sim}{B} \in \underset{\sim}{\mathfrak{F}}$, with $\underset{\sim}{B} \subset \underset{\sim}{A}$, $\underset{\sim}{\mu}(\underset{\sim}{B}) = \underset{\sim}{\mu}(\underset{\sim}{A})$ such that $f_n \xrightarrow{c} f$ on $\underset{\sim}{B}$, namely, $\underset{\sim}{B} \subset \chi_D$, where $D = \{f_n \to f\}$. For arbitrarily given $\varepsilon > 0$ and $x \in D$, there exists $N(x)$, such that $|f_n(x)-f(x)|<\varepsilon$, when $n \geqslant N(x)$. Denote $\underset{\sim}{A}_n = \underset{\sim}{B} \cap \chi_{\{x;\ N(x) \leqslant n\} \cap D}$, we have $\underset{\sim}{A}_n \nearrow \bigcup_{n=1}^{\infty} \underset{\sim}{A}_n = \underset{\sim}{B} \cap \chi_D = \underset{\sim}{B}$. Since $\underset{\sim}{A}_n \subset \underset{\sim}{A} \cap \chi_{\{|f_n - f| < \varepsilon\}}$,

$$\underset{\sim}{\mu}(\underset{\sim}{A}) \geqslant \underset{\sim}{\mu}(\underset{\sim}{A} \cap \chi_{\{|f_n - f| < \varepsilon\}}) \geqslant \underset{\sim}{\mu}(\underset{\sim}{A}_n) \to \underset{\sim}{\mu}(\underset{\sim}{B}) = \underset{\sim}{\mu}(\underset{\sim}{A}) ,$$

therefore $f_n \xrightarrow{F \cdot \underset{\sim}{\mu}} f$ on $\underset{\sim}{A}$.

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