ON IMPROPER INTEGRAL OF FUZZY MAPPING AND ON SERIES OF FUZZY NUMBERS

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In this paper the definitions of improper integrals and series of fuzzy numbers are introduced and other related objects are defined. Next their properties are presented. In spite of it that we have received results similar - in form - to the classical theory it will be clear to everybody after having read this paper that the presented theory is different from the theory of crisp improper integrals and series of real numbers. In this paper we will use the notions and definitions from [1], [2] and [3].

2. Improper integrals.

Recall that the definite integral is defined only for fuzzy mapping bounded on a finite interval. The definition of the definite integral breaks down if either the fuzzy mapping is unbounded or the interval is infinite. When this is the case, the definition of the integral is generalized by taking the integral over suitable finite intervals on which the fuzzy mapping is bounded and then considering the limit of those integrals. If the limit exists, the generalized integral is said to converge and if the limit does not exist the integral is said to diverge. Such integrals are called improper or infinite integrals.

Suppose F is integrable on $[a,b]$ for each $b \geq a$ and let $G(b) = \int_a^b F$ where $b \in [a,\infty)$. Then $\int_a^\infty F$ is called an improper (infinite) integral of the first kind.

We say that $\int_a^\infty F$ converges if $\lim_\infty G$ and in such a case the value $\int_a^\infty F$ is $\lim_\infty G$, i.e.,

$$\int_a^\infty F = \lim_{b\to\infty} G(b) = \lim_{b\to\infty} \int_a^b F .$$

If $\lim_\infty G$ does not exist, $\int_a^\infty F$ is said to diverge.

Suppose F is bounded and integrable on each interval $[a,c]$ where $c \in [a,b)$, but unbounded on $[a,b]$ and let $G(c) = \int_a^c F$ where $c \in [a,b)$. Then $\int_a^b F$ is called an improper integral of the second kind and the value of $\int_a^b F$ is $\lim_{b^-} G$ if this limit exists.

Thus

$$\int_a^b F = \lim_{c\to b^-} G(c) = \lim_{c\to b^-} \int_a^c F.$$

Let F be bounded and integrable on each interval $[c,d]$, where $c \in (a,b]$, but unbounded on $(a,b]$ and let $G(c) = \int_c^b F$ where $c \in (a,b]$.

Then $\int_a^b F$ is said to converge if $\lim_{a^+} G$ exists and in such a case the value of $\int_a^b F$ is $\lim_{a^+} G$, i.e.,

$$\int_a^b F = \lim_{c \to a^+} G(c) = \lim_{c \to a^+} \int_c^b F.$$

If $\lim_{a^+} G$ does not exist $\int_a^b F$ is said to diverge.

The improper integral $\int_{-\infty}^{\infty} F$ is defined to be $\int_{-\infty}^{a} F + \int_a^{\infty} F$ where a is any real number. If both of the integrals $\int_{-\infty}^{a} F$ and $\int_a^{\infty} F$ converge then $\int_{-\infty}^{\infty} F$ is said to converge and if either $\int_{-\infty}^{a} F$ or $\int_a^{\infty} F$ diverges then $\int_{-\infty}^{\infty} F$ is said to diverge.

Theorem 2.1. If F and G are bounded on $[a, \infty)$ and $\int_a^{\infty} F$ and $\int_a^{\infty} G$ both converge, then

(1) $\int_a^{\infty} (F+G)$ converges and $\int_a^{\infty} (F+G) = \int_a^{\infty} F + \int_a^{\infty} G$,

(2) $\int_a^{\infty} cF$ converges and $\int_a^{\infty} cF = c \cdot \int_a^{\infty} F$ for any constant c.

Proof. Since for any $b \in [a, \infty)$

$$\int_a^b (F+G) = \int_a^b F + \int_a^b G \qquad \text{(see [3])},$$

$$\lim_{b \to \infty} \int_a^b (F+G) = \lim_{b \to \infty} \int_a^b F + \lim_{b \to \infty} \int_a^b G = \int_a^{\infty} F + \int_a^{\infty} G,$$

(see [2]).

Thus $\int_a^{\infty} (F+G)$ converges and

$$\int_a^{\infty} (F+G) = \int_a^{\infty} F + \int_a^{\infty} G.$$

Also, since

$$\lim_{b \to \infty} \int_a^b cF = \lim_{b \to \infty} c \cdot \int_a^{\infty} F = c \cdot \lim_{b \to \infty} \int_a^b F = c \cdot \int_a^{\infty} F,$$

$\int_a^\infty cF$ converges and $\int_a^\infty cF = c \cdot \int_a^\infty F$.

Theorem 2.2. If F and G are continuous with respect to the metric D on $[a, \infty)$, $\bar{0} \le F(t) \le G(t)$ for all $t \in [a, \infty)$, and $\int_a^\infty F$ converges, then $\int_a^\infty G$ converges, where $\bar{0}$ is the fuzzy number such that for any t

$$\bar{0}(t) = \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $H_1(b) = \int_a^b F$ and $H_2(b) = \int_a^b G$. Since F and G are nonnegative fuzzy mappings, H_1 and H_2 are nondecreasing and for all $b \in [a, \infty)$

$$\bar{0} \le H_1(b) \le H_2(b) \le \int_a^\infty G.$$

Thus H_1 is a bounded monotonic fuzzy mapping, so $\lim_{\infty} H_1 = \int_a^\infty F$ exists.

Corollary. If F and G are continuous with respect to the metric D on $[a, \infty)$, $\bar{0} \le G(t) \le F(t)$ for all $t \in [a, \infty)$, and $\int_a^\infty G$ diverges, then $\int_a^\infty F$ diverges.

Proof. Suppose $\int_a^\infty F$ converges. Then, by the above Theorem $\int_a^\infty G$ converges contrary to hypothesis. This proves the corollary.

3. Series of fuzzy numbers.

Let $\{X_k\}$ be a sequence of fuzzy numbers, and $S_n = \sum_{k=1}^{n} X_k$. The sequence $\{S_n\}$ is called a series and the terms of $\{X_k\}$ are called the terms of the series.

We usually denote the series $\{S_n\}$ by $\sum_{k=1}^{\infty} X_k$ or, merely $\sum X_k$. If the sequence $\{S_n\}$ converges to the fuzzy number S, then we say that the series $\sum X_k$ has the sum S or $\sum X_k$ converges to S. The sum S_n of the first n terms of the series is sometimes called a partial sum. Thus our definition states that a series converges if and only if the sequence of partial sums converges. Since a sequence may or may not converge, a series may or not have a sum.

If the series $\sum_{k=1}^{\infty} X_k$ has the sum S, then $S = \lim S_n$ where $S_n = \sum_{k=1}^{n} X_k$. Hence $S = \lim_{n\to\infty} \sum_{k=1}^{n} X_k$. We will write $S = \sum_{k=1}^{\infty} X_k$.

We now give some properties of series of fuzzy numbers. Using the theory of sequences of fuzzy numbers, we obtain these properties of series from the corresponding properties of finite sums.

Theorem 3.1. If $\sum X_k$ and $\sum Y_k$ are convergent series with sums X and Y, respectively, and if c is a rela number, then

(1) $\sum (X_k+Y_k)$ converges to $X+Y$,

(2) $\sum cX_k$ converges to $c\cdot X$.

Proof. We prove (1) only. The proof of the second part is similar. Using the properties of the limits of the sequences of fuzzy numbers we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} (X_k+Y_k) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} X_k + \sum_{k=1}^{n} Y_k \right) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} X_k + \lim_{n\to\infty} \sum_{k=1}^{n} Y_k =$$

$$= X + Y.$$

According to the basic concepts, if we wish to determine whether the series $\sum X_k$ converges, all we need do is investigate the convergence of the sequence $\{S_n\}$ where $S_n = \sum_{k=1}^{n} X_k$. However, in many cases we are not able to obtain an expression for S_n from which we can determine the convergence of $\{S_n\}$. Thus, it is desirable to evalue criteria for the convergence of the series $\sum X_k$ in terms of the sequence $\{X_k\}$. The basic test for convergence and divergence of series with nonnegative terms is the comparison test.

Theorem 3.2. If $\sum X_k$ and $\sum Y_k$ are series with nonnegative terms, if $\sum Y_k$ converges, and if $X_k \leq Y_k$ for all k sufficiently large, then $\sum X_k$ converges.

Proof. Suppose $X_k \leq Y_k$ for all k greater than some positive integer N and $\sum Y_k = Y$. Then, for all $n > N$

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{N} X_k + \sum_{k=N+1}^{n} X_k \leq \sum_{k=1}^{N} X_k + \sum_{k=N+1}^{n} Y_k \leq$$

$$\leq \sum_{k=1}^{N} X_k + Y.$$

Since $Y \geq \bar{0}$, $S_n \leq \sum_{k=1}^{N} X_k + Y$ for all n and, hence, $\{S_n\}$ is bounded. Thus, $\sum X_k$ converges.

Corollary. If $\sum X_k$ and $\sum Y_k$ are series with nonnegative terms, if $\sum Y_k$ diverges, and if $X_k \geq Y_k$ for all k sufficiently large, then $\sum X_k$ diverges.

Proof. Suppose $\sum X_k$ converges. Then, by above Theorem, $\sum Y_k$ converges contrary to hypothesis. This proves the corollary.

We now give another convergence test for series of fuzzy numbers in which we compare a series with a corresponding improper integral.

Theorem (Integral Test). If $\sum X_k$ is a series with nonnegative terms and F is a nonincreasing continuous fuzzy mapping on the interval $[1, 1)$ such that $F(k) = X_k$, then $\sum X_k$ and $\int_1^{\infty} F$ either both converge or both diverge.

Proof. Recall that the improper integral $\int_1^\infty F$ is defined to be $\lim_{b\to\infty} \int_1^b F$. Thus, if we let $G(b) = \int_1^b F$, then $\int_1^\infty F = \lim_{b\to\infty} G(b)$. Since on $[1,\infty)$ the values of F are nonnegative, G is a nondecreasing fuzzy mapping.

(1) Assume $\int_1^\infty F$ converges to C. Since F is a nonincreasing fuzzy mapping, if k 2 then $X_k = F(k) \leqslant F(t)$ for $t \in [k-1, t]$ and, therefore,

$$X_k = \int_{k-1}^{k} X_k \leqslant \int_{k-1}^{k} F.$$

Then

$$S_n = \sum_{k=1}^{n} X_k \leqslant a_1 + \sum_{k=2}^{n} \int_{k-1}^{k} F = a_1 + \int_1^n F \leqslant X_1 + C.$$

Thus, $\{S_n\}$ is a bounded nondecreasing sequence and, hence, $\sum X_k$ converges.

(2) Assume $\int_1^\infty F$ diverges. Since

$$X_k = F(k) \geqslant F(t) \quad \text{for } t \in [k, k+1],$$

$$X_k = \int_k^{k+1} X_k \geqslant \int_k^{k+1} F.$$

Then

$$S_n = \sum_{k=1}^{n} X_k \geqslant \sum_{k=1}^{n} \int_k^{k+1} F = \int_1^{n+1} F$$

and, since $\int_1^\infty F$ diverges, the sequence $\{S_n\}$ diverges. Therefore $\sum X_k$ diverges. This completes the proof.

References

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2. M. Matłoka, Limits and continuity of the fuzzy functions, BUSEFAL (in print).

3. M. Matłoka, On integral of fuzzy mapping, BUSEFAL (in print).