FUZZY DIFFERENTIAL EQUATIONS

Yunqing Wang and Shixian Wu

Dept. of Mathematics, Hebei Teachers' University,
Shijiazhuang, Hebei, P.R.China.

This paper introduces the idea of fuzzy differential equations. We show some theorems for fuzzy functions. We present solutions for linear fuzzy differential equations With real functions coefficients.

Keywords: Fuzzy functions, Derivative of fuzzy functions, Fuzzy differential equations.

1. Fuzzy functions and its derivatives

Definition of a fuzzy number is given in [1]. We let $\mathcal{F}$ denote the family of fuzzy numbers. A real number is determined by a fuzzy number

$$S_a = \begin{cases} 1, & \text{if } x=a \\ 0, & \text{if } x \neq a \end{cases}$$

We can identify a fuzzy number $\tilde{\mu}$ with the parameterized triples

$$\{(a(r), b(r), r) \mid 0 \leqslant r \leqslant 1\} \qquad \cdots\cdots\cdots\cdots \quad (1)$$

where $a(r)$ denotes the left hand endpoint of $c_r(\tilde{\mu})$ and $b(r)$ denotes the right hand endpoint.

And
$$c_r(\tilde{\mu}) = \begin{cases} \{x \mid \tilde{\mu}(x) \geqslant r\} & \text{if } 0 < r \leqslant 1, \\ cl(\operatorname{supp} \tilde{\mu}) & \text{if } r=0, \end{cases}$$

where $cl(\operatorname{supp}\tilde{\mu})$ denotes the closure of the support of $\tilde{\mu}$.

If $\tilde{\mu}: R \to I$ is a fuzzy number with parameterization given by (1), then the functions a and b satisfy five conditions in [1]. Moreover , suppose that $a: I \to R$ and $b: I \to R$ satisfy the five conditions in [1], then $\tilde{\mu}: R \to I$ defined by

$$\tilde{\mu}(x) = \sup\{r \mid a(r) \leqslant x \leqslant b(r)\} \qquad \cdots\cdots\cdots\cdots \quad (2)$$

is a fuzzy number with parameterzation given by (1).

Let $\mathcal{V} = \{\{(a(r), b(r), r) \mid r \in I\} \mid a: I \to R,\ b: I \to R \text{ are bounded functions.}\}$

We define the addition, the scalar product and the metric on $\mathcal{V}$ by (3), (4) and (5).

$$\{(a(r), b(r), r) \mid r \in I\} + \{(c(r), d(r), r) \mid r \in I\}$$
$$= \{(a(r)+c(r), b(r)+d(r), r) \mid r \in I\} \qquad \cdots\cdots \quad (3)$$

$$c\{a(r), b(r), r) \mid r \in I\} = \{(ca(r), cb(r), r) \mid r \in I\} \qquad \cdots\cdots \quad (4)$$

$$D\left(\{(a(r),b(r),r) \mid r\in I\},\ \{(c(r),d(r),r \mid r\in I\}\right)$$
$$=\sup\left\{\max\left\{|a(r)-c(r)|,|b(r)-d(r)|\right\} r\in I\right\} \quad \ldots\ldots \quad (5)$$

It is clear that the vector space $\mathcal{V}$ together with the metric form a topological vector space.

A function $\tilde{f}:R\to\mathcal{F}$ is said to be a fuzzy function.

Limit and continuouty of fuzzy functions are studied with respect to the metric D defined by (5).

Theorem 1. Fuzzy function $\tilde{f}(x) = \{(a(r,x),b(r,x),r) \mid r\in I\}$ is continuous at $x_0 \longleftrightarrow$ The families $\{a(r,x) \mid r\in I\}$ and $\{b(r,x) \mid r\in I\}$ are equicontinuous with respect to x at x_o.

Proof. $\tilde{f}(x)$ is continuous at x_o

$\longleftrightarrow$ for any given $\varepsilon>0$ there exists a $\delta>0$ such that if

$|x-x_0|<\delta$, then $D(\tilde{f}(x),\tilde{f}(x_0))<\varepsilon$.

$\longleftrightarrow \forall\varepsilon>0, \exists\delta>0$, such that if $|x-x_0|<\delta$, then

$$\left.\begin{array}{l}|a(r,x)-a(r,x\ )|<\varepsilon\\ |b(r,x)-b(r,x\ )|<\varepsilon,\end{array}\right\} \text{for any } r\in I.$$

$\longleftrightarrow$ families $\{a(r,x)|r\in I\}$ and $\{b(r,x)|r\in I\}$ are equicontinuous with respect to x at x_o.

Definition. Suppose that $\tilde{f}:R\to\mathcal{F}$ and let $x_o\in R$. The derivative $\tilde{f}'(x_o)$ of $\tilde{f}$ at the point x_o is defined by

$$\tilde{f}'(x_o) = \lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$$

provided this limit exists.

There are two results for fuzzy derivative in [1].

(I) Suppose that $\tilde{f}:R\to\mathcal{F}$, $x_o\in R$ and that

$$\lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$$

exists. Let $\eta=\{(\alpha(r),\beta(r),r) \mid r\in I\}$ be the parametric representation of this limit and for each $x\in R$, let $\{(a(r,x),b(r,x),r)|r\in I\}$ be the parametric representation of $\tilde{f}(x)$. Then

(i) $\eta\in\mathcal{V}$, and

(ii) $\alpha(r)=a_x(r,x_o)$ and $\beta(r)=b_x(r,x_o)$ for each $r\in I$

where a_x and b_x are the partial derivatives of a and b with respect to x.

(II) Suppose that $\tilde{f}:R\to\mathcal{F}$ is a fuzzy function and that for each x, $\tilde{f}(x)$ is represented parametrically by $\{(a(r,x),\ b(r,x),r)\mid r\in I\}$. If a_x and b_x are continuous, then $\tilde{f}'(x)$ exists for each $x\in R$.

The proof of following theorem is straight forward (and omitted)

Theorem 2. Suppose that $\tilde{f}(x)=\{(a(r,x),b(r,x),r)\mid r\in I\}$ is nth differentiable fuzzy function, g(x) is nth differentiable real function. Then

$$(\tilde{f}\cdot g)^{(n)}=\sum_{k=0}^{n}C_n^k\ \tilde{f}^{(n-k)}g^{(k)}$$

This result is similar to Leibniz formula for distribution. Thus in this paper we only discuss fuzzy differential equations with real functions coefficients.

2. Fuzzy differential equations

First we consider 1st order linear fuzzy differential equation

$$\tilde{f}'(x)+p(x).\tilde{f}(x)=\tilde{g}(x)\qquad\ldots\ldots\qquad(1)$$

where $\tilde{f}(x)$ is a unknown fuzzy function with continuous derivative, $\tilde{g}(x)$ is a known continuous fuzzy function, p(x) is a known real function. Suppose that

$$\tilde{f}(x)=\{(\alpha(r,x),\ \beta(r,x),r)\mid r\in I\}\qquad\ldots\ldots\qquad(2)$$
$$\tilde{g}(x)=\{(a(r,x),\ b(r,x),r)\mid r\in I\}$$

By (1) and (I) we have that

$$\alpha_x(r,x)+p(x)\,\alpha(r,x)=a(r,x)\qquad\ldots\ldots\qquad(1.1)$$
$$\beta_x(r,x)+p(x)\,\beta(r,x)=b(r,x)\qquad\ldots\ldots\qquad(1.2)$$

Their solutions are

$$\alpha(r,x)=e^{-\int_{x_0}^{x}p(t)dt}\left(\int_{x_0}^{x}a(r,t)e^{\int_{x_0}^{t}p(\xi)d\xi}\,dt+c_1(r)\right)\qquad\ldots\ldots\qquad(1.3)$$

$$\beta(r,x)=e^{-\int_{x_0}^{x}p(t)dt}\left(\int_{x_0}^{x}b(r,t)e^{\int_{x_0}^{t}p(\xi)d\xi}\,dt+c_2(r)\right)\qquad\ldots\ldots\qquad(1.4)$$

where x_0 is constant, $c_1(r)$, $c_2(r)$ are intergral constants. We can prove that $\tilde{f}(x)=\{(\alpha(r,x),\ (r,x),r)\mid r\in I\}$ is a fuzzy function, where $\alpha(r,x)$ and $\beta(r,x)$ are given by (1.3) and (1.4)respectively, $\{(c_1(r),c_2(r),r)\mid r\in I\}$ is a fuzzy number. $\tilde{f}(x)$ is general solution of the equation (1).

The initial-value problem of fuzzy differential equation

$$(A)\quad \begin{cases} \tilde{f}'(x) + p(x)f(x) = g(x) \\ \tilde{f}(x)\big|_{x=x_0} = \tilde{\mu} = \{(d(r), e(r), r) \mid r \in I\} \end{cases}$$

can be changed into the initial-value problem of two ordinary differential equations

$$(A\text{-}1)\quad \begin{cases} \alpha_x(r,x) + p(x)\alpha(r,x) = a(r,x) \\ \alpha(r,x_0) = d(r) \end{cases}$$

$$(A\text{-}2)\quad \begin{cases} \beta_x(r,x) + p(x)\ (r,x) = b(r,x) \\ \beta(r,x_0) = e(r) \end{cases}$$

The solution of (A-1) is

$$\alpha(r,x) = e^{-\int_{x_0}^{x} p(t)dt}\left(\int_{x_0}^{x} a(r,t)e^{\int_{x_0}^{t} p(\xi)d\xi}\,dt + d(r)\right)$$

and solution of (A-2) is

$$\beta(r,x) = e^{-\int_{x_0}^{x} p(t)dt}\left(\int_{x_0}^{x} b(r,t)e^{\int_{x_0}^{t} p(\xi)d\xi}\,dt + e(r)\right)$$

Now we put above two expressions into (2). The solution of (A) is obtained.

Example 1. Find the solution of the following initial-value problem of fuzzy differential equation

$$\begin{cases} \tilde{f}'(x) + \frac{1}{50+x}\tilde{f}(x) = S_{24} \\ \tilde{f}(0) = S_{25} \end{cases}$$

Since $S_{24}=\{(24,24,r) \mid r \in I\}$, $S_{25}=\{(25,25,r) \mid r \in I\}$

$$\int p(x)dx = \int \frac{dx}{50+x} = \ln(50+x)$$

$$\int a(r,x)e^{\int p(x)dx}dx = \int 24(50+x)dx = 1200x + 12x^2$$

Thus $\alpha(r,x) = \dfrac{c_1(r)+1200x+12x^2}{50+x}$

$$\beta(r,x) = \frac{c_2(r)+1200x+12x^2}{50+x}$$

Since $\alpha(r,0)=25$ thus $c_1(r)=1250$. Similarly $c_2(r)=1250$.

The solution of (A) is

$$\tilde{f}(x)=\left\{\left(\frac{12x^2+1200x+1250}{50+x}, \frac{12x^2+1200x+1250}{50+x}, r\right) \,\middle|\, r \in I\right\}$$

$$= S_{\frac{12x^2+1200x+1250}{50+x}}$$

The example shows that ordinary differential equations are particular case of fuzzy differential equations.

Example 2. Find the solution of the fuzzy differential equation

$$\tilde{f}'(x) = \tilde{v}$$

where

$$\tilde{v} = \begin{cases} 1-(t-a)^2, & \text{if } t\in[a-1,a+1], \\ 0, & \text{if } t\bar{\in}[a-1,a+1], \end{cases}$$

where a is a real number.

It is obvious that $\tilde{v}=\{(a-\sqrt{1-r},\ a+\sqrt{1-r},r) \mid r\in I\}$

Let $\tilde{f}(x)=\{(\alpha(r,x),\beta(r,x),r) \mid r\in I\}$

We obtain two ordinary differential equations

$$\alpha_x(r,x)= a-\sqrt{1-r} \quad , \quad \beta_x(r,x)= a+\sqrt{1-r}$$

Thus $\alpha(r,x)=(a-\sqrt{1-r})x+ c_1(r), \ \beta(r,x)=(a+\sqrt{1-r})x+ c_2(r)$

Hence

$$\tilde{f}(x)=\{((a-\sqrt{1-r})x+c_1(r),(a+\sqrt{1-r})x+c_2(r),r) \mid r\in I\} = x\tilde{v} + \tilde{\mu}$$

Where $\mu = \{(c_1(r),c_2(r),r) \mid r\in I\}$. The $\tilde{\mu}$ is arbitrary fuzzy constant.

The solution of the initial-value problem $\begin{cases}\tilde{f}'(x)=\tilde{v}\\ \tilde{f}(0)=0\end{cases}$ is $\tilde{f}(x)=x\tilde{v}$.

Readers can easily see that the solution represents a motion of the particle with fuzzy number speed $\tilde{v}$.

To sum up, we can conclude that finding solutions to a fuzzy differential equations with real function coefficient are actually solving two ordinary differential equations. Thus 2ed until nth order linear fuzzy differential equations with constant coefficient can be theoretically solved.

References

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