FUZZY DIFFERENTIAL EQUATIONS
A problem letter

Fuzzy functions and their derivatives. After defining the notion of fuzzy set, one pays attention to the notions of fuzzy function and its derivative. Consider, for example, fuzzy-valued functions of time, i.e. abstract functions of the scalar argument and with fuzzy sets as values [1]. If the space of these values is isomorphic to a cone of a normed space, then for such functions the notion of conical derivative can be introduced [2]. That will be another fuzzy-valued function of time. Conical differentiability is rather strong requirement: it enables the irreversibility of time to be corroborated mathematically [3]. In turn, after introducing the notion of conical derivative, one can consider formal differential equations with fuzzy-valued functions of time as solutions, i.e. ordinary differential equations in the space of fuzzy sets. In the case of the single-point (concentrated) fuzzy sets all is supposed to be reduced to ordinary differential equations in the original space of points.

<u>Evolution of uncertainty</u>. Let there be an object with an initial state and a dynamics both inexactly known. How will the uncertainty concerning its state change in the course of time? Whether this uncertainty ought to increase? But if 'uncertainty draining', including any processes of real observation and measurement, is absent? In the case of objective uncertainty, such questions are discussed within the framework of entropy analysis (here I would like to point out the works [4;5]). And when uncertainty is subjective, it is natural to apply ideas related to fuzzy sets; dynamical systems under fuzzy feedback would be a good example for the mentioned solutions to describe their state evolution with time. However natural this suggestion is, I found that ordinary differential equations in the space of fuzzy sets were not directly studied. Perhaps it happened so because the idea of the correct Cauchy problem for ordinary differential equations with <u>fuzzy initial condition</u> seems trivial at first sight. Indeed, one can take the solution $\tau \to x_\tau(x_0)$ of $dx/d\tau = f_\tau(x)$ for any initial state x_0, put a fuzzy initial state μ_0 in place of x_0 according to Zadeh's extension principle, and consider the obtained function $\tau \to x_\tau(\mu_0)$ as a solution of

$dx/d\tau = f_\tau(x)$ for an initial state fuzzified by μ_0. The solution so defined has the following feature: the initial fuzziness, included in μ_0, is transfered (preserved) along with the trajectories $x_\cdot(x_0)$, i.e. the value of the characteristic function of the fuzzy set $x_\tau(\mu_0)$ at the point $x_\tau(x_0)$ does not depend on τ. But that is the main — whether the initial fuzziness ought to be preserved along with the trajectories! Also thinkable are other ways of the solution defining. For example, there exists the definition that leads the initial fuzziness to grow along with the trajectories [6].

<u>Differential equations with set-valued solutions, differential inclusions, methods of interval analysis.</u> So it is possible to speak only of adjacent results. Here I hint largely at ordinary differential equations in the space of ordinary sets or, that is the same, differential equations with set-valued solutions [7]. It is necessary to distinguish them from the so-called differential inclusions (other names: multivalued differential equations, differential equations with multivalued right-hand side, generalized differential equations) and from contingent and paratingent equations being close to differential inclusions [8; 12], especially in connection with the notion of differential

inclusion 'integral funnel' [14; 15]. By means of the set-valued solutions it is possible to bound the solution families of differential inclusions (see Appendix). Besides, to evaluate the solution of that Cauchy problem in the space R^1 whose the initial condition and right-hand side values are set only within certain intervals (segments), methods of interval analysis are utilized [13]; as regards R^n, n-dimensional rectangles figure instead of intervals [9; p. 93-99].

<u>The major problem.</u> Extremely important stays the problem of knowledge, namely what is the status of observation, measurement, a priori knowledge, etc. — and how is all this to be included in the theory? For example, one may not know where the object is moving, but he may reduce his unknowledge by observing. What happens to subjective uncertainty in measuring? How does 'fuzziness draining' flow? The point is that the mentioned dynamical systems under fuzzy feedback indicate a situation in which there are 'sources' of fuzziness but are not its 'drains', so the fuzziness has to increase. Adequate to such a situation is Minkowski's addition, which is not supposed to have the inverse operation. This is the simplest solution to the problem of drainage / subtraction.

Appendix. Let n be a positive integer. By R^n, as well as earlier on, we mean the n-dimensional arithmetical space with the Euclidean norm $|\cdot|$, and by Ω the space of all non-empty convex compact subsets of R^n — with the Minkowski addition $+$, the positive homothetic and the Hausdorff metric (see [10]). Let

$$x \in R^n,\ \tau \in T = [\tau_0, \tau_1] \subset R^1,\ M \in \Omega;\ S = \{x : |x| \leq 1\}.$$

Let $F_\cdot(\cdot) : T \times R^n \to \Omega$ be a continuous in τ and Lipschitzian in x mapping. Under these circumstances it can be proven (by the method of successive approximations) that if $x_\cdot$ is such a differentiable function from T to R^n that

$$dx_\tau / d\tau \in F_\tau(x_\tau) \quad \forall \tau,$$

and if $M_\cdot$ is such a conically differentiable function from T to Ω (see [11]) that

$$dM_\tau / d\tau = \mathrm{co} \bigcup_{x \in M_\tau} F_\tau(x) \quad \forall \tau$$

(where the left-hand side denotes the conical derivative of $M_\cdot$ at an instant τ, and the other side is the convexified union of the sets), then

$$x_{\tau_0} \in M_{\tau_0} \implies x_\tau \in M_\tau \quad \forall \tau.$$

For example, if

$$|dx_\tau / d\tau - x_\tau| \leq 1 \quad \forall \tau$$

and if $x_{\tau_0} \in M$, then

$$x_\tau \in e^{\tau - \tau_0} M + (e^{\tau - \tau_0} - 1) S \quad \forall \tau.$$

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