ON INTEGRAL OF FUZZY MAPPINGS

Marian Matłoka

Institute of Economical Cybernetics, Department of Mathematics,

Economic Academy of Poznań, ul Marchlewskiego 146/150,

60-967 Poznań, Poland

1. Introduction. The aim of the present paper is to explain new concept of an integral of fuzzy mapping. Such integral is generalization of the Riemann integral over a closed interval to fuzzy mappings. The interpretation is quite straightforward: such an integral can be used to evaluate the surface of an area delimited by an ill-defined borderline. In this paper we will use the notions and definitions from [1].

2. Preliminaries.

A set $\mathcal{D}$ of fuzzy numbers from L(R) (see [1]) is bounded above (or has an upper bound) if there exists a fuzzy number C such that, for all $X \in \mathcal{D}$, $X \leqslant C$ (see [1]). Any such fuzzy number C is called an upper bound of $\mathcal{D}$.

A set $\mathcal{D}$ of fuzzy numbers from L(R) is bounded below (or has a lower bound) if there exists a fuzzy number C such that, for all $X \in \mathcal{D}$, $X \geqslant C$. Any such fuzzy number C is called a lower bound of $\mathcal{D}$.

A fuzzy number C is called the least upper bound of a set $\mathcal{D}$, written $C = \text{lub}\,\mathcal{D}$, if C is an upper bound of $\mathcal{D}$ and no fuzzy number smaller than C is an upper bound of $\mathcal{D}$.

A fuzzy number C is called the greatest lower bound of a set $\mathcal{D}$, written $\text{glb}\,\mathcal{D}$, if C is a lower bound of $\mathcal{D}$ and no fuzzy number greater than C is a lower bound of $\mathcal{D}$.

A fuzzy mapping F from a set $T \subset R$ to a set L(R) is a mapping from T to the set of all fuzzy numbers $X \in L(R)$.

In other words, to each number $t \in T$ corresponds a fuzzy number F(t) from L(R).

A fuzzy mapping F is said to be bounded above, bounded below on a set D contained in the domain of the fuzzy mapping if the set of mapping values, $\{F(t) : t \in D\}$, is bounded above, bounded below respectively.

If a fuzzy mapping F is bounded above and bounded below then we will say that it is bounded.

From the above definitions it follows that if F is bounded then there exist fuzzy numbers L and U such that $L \leq F(t) \leq U$ for all $t \in D$.

3. Integral of fuzzy mapping.

Let F be a fuzzy mapping whose domain of definition includes the closed interval [a,b]. The symbol for "the integral of fuzzy mapping F from a to b" is $\int_a^b F$.

Definition 3.1. A finite set of numbers $t_0, t_1, \ldots, t_n$ is said to be a partition of [a,b] if $a = t_0 \leq t_1 \leq \ldots \leq t_n = b$.
We denote a partition of [a,b] by $P = \{t_i : i = 0,1,\ldots,n\}$, and define the mesh of P, written $|P|$, by

$$|P| = \max\{t_j - t_{j-1} : j=1,\ldots,n\}.$$

The mesh of a partition is a measure of the fineness of the partition.

Let F be a fuzzy mapping bounded on [a,b], i.e., there are fuzzy numbers L and U such that $L \leq F(t) \leq U$ for all $t \in [a,b]$. Define

$$U_i(F) = \text{lub}\{F(t) : t \in [t_{i-1}, t_i]\},$$

$$L_i(F) = \text{glb}\{F(t) : t \in [t_{i-1}, t_i]\}.$$

Since F is assumed to be bounded, $U_i(F)$ and $L_i(F)$ exist for each $i=1,\ldots,n$, and

$$L \leq L_i(F) \leq U_i(F) \leq U \qquad (*)$$

The definition of integral will be given in terms of sums of the following types:

$$L(F,P) = \sum_{i=1}^{n} L_i(F)\cdot(t_i - t_{i-1})$$

and

$$U(F,P) = \sum_{i=1}^{n} U_i(F)\cdot(t_i - t_{i-1}).$$

We call L(F,P) the "lower sum", and U(F,P) the "upper sum", corresponding to the partition P.

By (*)

$$L\cdot(b-a) = \sum_{i=1}^{n} L\cdot(t_i - t_{i-1}) \leq \sum_{i=1}^{n} L_i(F)\cdot(t_i - t_{i-1}) \leq$$

$$\leq \sum_{i=1}^{n} U_i(F)\cdot(t_i - t_{i-1}) \leq \sum_{i=1}^{n} U\cdot(t_i - t_{i-1}) =$$

$$= U\,(b-a)$$

or

$$L\cdot(b-a) \leq L(F,P) \leq U(F,P) \leq U\cdot(b-a) \qquad (***)$$

Let $\mathcal{P}$ be the set of all partitions of the interval $[a,b]$. The inequality (***) holds for each partition P in $\mathcal{P}$ and tells us that set of fuzzy numbers $\{L(F,P) : P \in \mathcal{P}\}$ – the set of all lower sums obtained by taking all possible partitionsof $[a,b]$ – has an upper bound, namely, $U\cdot(b-a)$. The set $\{L(F,P) : P \in \mathcal{P}\}$ has, therefore, a least upper bound. Similarly, the set $\{U(F,P) : P \in \mathcal{P}\}$ of all upper sums has a lower bound $L\cdot(b-a)$, so that $\{U(F,P) : P \in \mathcal{P}\}$ has a greatest lower bound. This least upper bound and this greatest lower bound are sufficiently important that we introduce names and symbols for them.

Definition 3.2. Define

$$\underline{\int}_a^b F = \text{lub}\{L(F,P) : P \in \mathcal{P}\}$$

and

$$\overline{\int}_a^b F = \text{glb}\{U(F,P) : P \in \mathcal{P}\}.$$

$\underline{\int}_a^b F$ is called the lower integral of F from a to b, and $\overline{\int}_a^b F$ is called the upper integral of F from a to b.

If $P = \{t_i : i=0,\dots,n\}$ and $P' = \{t_i' : i=0,\dots,n\}$ are partitions of an interval $[a,b]$, $P \subset P'$ means that each division point t_i of P is also a division point of P'. When this is the case, P' is said to be a "refinement" of P. We now show that a refinement of a partition does not decrease the lower sum nor increase the upper sum. Stated precisely, we prove for a bounded fuzzy mapping F that:

Lemma 3.1. $P \subset P'$ implies $L(F,P) \leq L(F,P')$ and $U(F,P') \leq U(F,P)$.

Proof. If $P = P'$, the lemma is obviously true. Assume that $P \neq P'$ and $P \subset P'$. Let t_j' be the first division point of P' that is not in P. Then, for some k, $t_{k-1} \leq t_j' \leq t_k$. Define

$$P_1 = \{t_0, t_1, \dots, t_{k-1}, t_j', t_k, \dots, t_n\},$$

$$L^*(F) = \mathrm{glb}\{F(t) : t \in [t_{k-1}, t_j']\},$$

and

$$L^{**}(F) = \mathrm{glb}\{F(t) : t \in [t_j', t_k]\}.$$

From the definition of $L_k(F)$, $L_k(F) \leq L^*(F)$ and $L_k(F) \leq L^{**}(F)$.

Therefore

$$L_k(F)\cdot(t_k - t_{k-1}) = L_k(F)\cdot(t_j' - t_{k-1}) + L_k(F)\cdot(t_k - t_j') \leq$$
$$\leq L^*(F)\cdot(t_j' - t_{k-1}) + L^{**}(F)\cdot(t_k - t_j'),$$

and hence

$$L(F,P) = \sum_{i=1}^{n} L_i(F)\cdot(t_i - t_{i-1}) \leq L_1(F)\cdot(t_1 - t_0) +$$
$$+ L_2(F)\cdot(t_2 - t_1) + \dots + L_{k-1}(F)\cdot(t_{k-1} - t_{k-2}) +$$
$$+ L^*(F)\cdot(t_j' - t_{k-1}) + L^{**}(F)\cdot(t_k - t_j') +$$
$$+ L_{k+1}(F)\cdot(t_{k+1} - t_k) + \dots + L_n(F)\cdot(t_n - t_{n-1}) =$$
$$= L(F,P_1).$$

Repeating this procedure a finite number of times, we can add to P all of t_i' of P' that are not in P and thus obtain $L(F,P) \leq L(F,P')$. In a similar manner-the inequalities throughout are reversed - we obtain $U(F,P) \geqslant U(F,P')$.

Corollary. Let $\mathcal{P}$ be a set of all partitions P of a,b . Then from Lemma 3.1 we have

$$\lim_{P\in\mathcal{P}} U(F,P) = \bar{\int_a^b} F \,, \quad \lim_{P\in\mathcal{P}} L(F,P) = \underline{\int_a^b} F \,. \qquad \text{(see [1])}.$$

Lemma 3.2. If F is bounded on [a,b] and $L \leq F(t) \leq U$ for all $t \in [a,b]$, then

$$L\cdot(b-a) \leq \underline{\int_a^b} F \leq \bar{\int_a^b} F \leq U\cdot(b-a).$$

Proof. Since $\underline{\int_a^b} F$ is the least upper bound of $\{L(F,P) : P \in \mathcal{P}\}$, it follows that $L\cdot(b-a) \leq \underline{\int_a^b} F$. Similarly, $\bar{\int_a^b} F \leq U\cdot(b-a)$. It then remains to show that the least upper bound $\underline{\int_a^b} F$ of the lower sums cannot exceed the greatest lower bound $\bar{\int_a^b} F$ of the upper sums.

Let P_1 and P_2 be any pair of partitions of $[a,b]$, and define $P' = P_1 \cup P_2$. Now $P_1 \subset P'$ and $P_2 \subset P'$, and therefore P' is a refinement of both P_1 and P_2 . Then, by Lemma 3.1

$$L(F,P_1) \leq L(F,P') \quad \text{and} \quad U(F,P') \leq U(F,P_2).$$

From (***), which holds for an arbitrary partition P, we obtain

$$L(F,P_1) \leq L(F,P') \leq U(F,P') \leq U(F,P_2)$$

or

$$L(F,P_1) \leq U(F,P_2)$$

for each pair of partitions P_1 and P_2 of $[a,b]$. It then follows for each P_2 that $U(F,P_2)$ is an upper bound of $\{L(F,P_1) : P_1 \in \mathcal{P}\}$.

$$\underline{\int_a^b} F \leq U(F,P_2) \quad \text{for each } P_2 \in \mathcal{P} \,,$$

and $\underline{\int_a^b} F$ is a lower bound of $\{U(F,P_2) : P_2 \in \mathcal{P}\}$. Since $\bar{\int_a^b} F$ is the greatest lower bound of $\{U(F,P_2) : P_2 \in \mathcal{P}\}$, $\underline{\int_a^b} F \leq \bar{\int_a^b} F$.

This completes the proof.

We are particularly interested in those fuzzy mappings for which $\underline{\int}_a^b F = \overline{\int}_a^b F$. In this case the upper and lower sums approach each other and associate a unique fuzzy number with the fuzzy mapping F on $[a,b]$. This fuzzy number is the definite fuzzy integral.

Definition 3.3. A fuzzy mapping F on $[a,b]$ is said to be integrable on $[a,b]$ if F is bounded on $[a,b]$ and

$$\underline{\int}_a^b F = \overline{\int}_a^b F .$$

If F is integrable on $[a,b]$ then the definite integral of F from a to b, written $\int_a^b F$, is defined by

$$\int_a^b F = \underline{\int}_a^b F = \overline{\int}_a^b F .$$

Our definition of the integral of fuzzy mapping includes the case a=b. The interval $[a,a]$ consists of the single point a, and the only partition of $[a,a]$ is $P = \{a=t_0 , t_1=a\}$. Hence, if F is defined at a, then $L(F,P) = U(F,P) = F(a)\cdot(t_1 - t_0) = \bar{0}$. Therefore $\int_a^a F = \bar{0}$ for each fuzzy mapping F defined at a, where $\bar{0}$ is the fuzzy number such that for any $t \in R$

$$\bar{0}(t)=\begin{cases} 1 & \text{if } t=0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.1. If F is integrable on $[a,b]$ and P is any partition of $[a,b]$ then

$$L\cdot(b-a) \leq L(F,P) \leq \int_a^b F \leq U(F,P) \leq U\cdot(b-a).$$

Proof. By (**) and Definition 3.2, we have for any partition P of $[a,b]$,

$$L\cdot(b-a) \leq L(F,P) \leq \underline{\int}_a^b F$$

and

$$\int_a^b F \leq U(F,P) \leq U\cdot(b-a).$$

Now, F integrable on [a,b] implies $\int_{-a}^{b} F = \int_a^{\bar{b}} F$, and hence

$$L\cdot(b-a) \leq L(F,P) \leq \int_{-a}^{b} F = \int_a^b F = \int_a^{\bar{b}} F \leq U(F,P) \leq U\cdot(b-a).$$

Theorem 3.2. A bounded fuzzy mapping F is integrable on [a,b] if and only if corresponding to each $\varepsilon > 0$ there is a partition P with the property that $D(U(F,P),L(F,P)) < \varepsilon$.

Proof. Assume that corresponding to each $\varepsilon > 0$ such a P exists. Then, since

$$D(\int_a^{\bar{b}} F, \int_{-a}^{b} F) \leq D(U(F,P),L(F,P)) < \varepsilon \quad ,$$

it follows that $\int_a^{\bar{b}} F = \int_{-a}^{b} F$, and F is integrable on [a,b]. Conversely assume that F is integrable on [a,b] . Then $\int_{-a}^{b} F = \int_a^{\bar{b}} F = \int_a^b F$.

Therefore $\int_a^b F$ is the least upper bound of the lower sums $\{L(F,P)\}$ and is the greatest lower bound of the upper sums $\{U(F,P)\}$. Hence corresponding to each $\varepsilon > 0$ there is a P_1 such that

$$D(\int_a^b F, L(F,P_1)) < \varepsilon/2$$

and a P_2 such that

$$D(U(F,P_2), \int_a^b F) < \varepsilon/2.$$

So,

$$D(U(F,P_2),L(F,P_1)) \leq D(U(F,P_2), \int_a^b F) + D(L(F,P_1), \int_a^b F) < \varepsilon \ .$$

Let $P = P_1 \cup P_2$, P is a refinement of both P_1 and P_2. Therefore

$$D(U(F,P),L(F,P)) \leq D(U(F,P_2),L(F,P_1)) < \varepsilon \ .$$

This completes the proof.

4. The existence of integrable fuzzy mappings.

Theorem 4.1. Every fuzzy mapping continuous with respect to the metric D on $[a,b]$ is integrable on $[a,b]$.

Proof. Since F is continuous on $[a,b]$, that F is bounded on $[a,b]$. This means that corresponding to each partition P of $[a,b]$ for any $\alpha\in[0,1]$ there are the $t_i', t_i'^{*}\in[t_{i-1},t_i]$ and the $t_i'', t_i''^{*}\in[t_{i-1},t_i]$ such that

$$\underline{F(t_i')}^{\alpha} = \underline{L_i(F)}^{\alpha} \quad\text{and}\quad \overline{F(t_i'^{*})}^{\alpha} = \overline{L_i(F)}^{\alpha}$$

and

$$\underline{F(t_i'')}^{\alpha} = \underline{U_i(F)}^{\alpha} \quad\text{and}\quad \overline{F(t_i''^{*})}^{\alpha} = \overline{U_i(F)}^{\alpha}.$$

Now

$$d(U(F,P)^{\alpha}, L(F,P)^{\alpha}) = \max\left(|\underline{U(F,P)}^{\alpha}-\underline{L(F,P)}^{\alpha}|,\ |\overline{U(F,P)}^{\alpha}-\overline{L(F,P)}^{\alpha}|\right)$$

$$= \max\Big(\Big|\sum_{i=1}^{n}\underline{U_i(F)}^{\alpha}(t_i-t_{i-1}) - \sum_{i=1}^{n}\underline{L_i(F)}^{\alpha}(t_i-t_{i-1})\Big|,$$

$$\Big|\sum_{i=1}^{n}\overline{U_i(F)}^{\alpha}(t_i-t_{i-1}) - \sum_{i=1}^{n}\overline{L_i(F)}^{\alpha}(t_i-t_{i-1})\Big|\Big) =$$

$$= \max\Big(\Big|\sum_{i=1}^{n}\underline{F(t_i'')}^{\alpha}(t_i-t_{i-1}) - \sum_{i=1}^{n}\underline{F(t_i')}^{\alpha}(t_i-t_{i-1})\Big|,$$

$$\Big|\sum_{i=1}^{n}\overline{F(t_i''^{*})}^{\alpha}(t_i-t_{i-1}) - \sum_{i=1}^{n}\overline{F(t_i'^{*})}^{\alpha}(t_i-t_{i-1})\Big|\Big)$$

Continuity of F on $[a,b]$ implies that corresponding to each $\varepsilon>0$ there is $\delta>0$ such that $r,s\in[a,b]$ and $|r-s|<\delta$ imply $D(F(r),F(s)) < \varepsilon/(b-a)$. Selecting any partition P with $|P|<\delta$ – and such partitions do exist–, we then see that

$$|\underline{F(t_i'')}^{\alpha} - \underline{F(t_i')}^{\alpha}| < \varepsilon'/(b-a)$$

and

$$|\overline{F(t_i''^{*})}^{\alpha} - \overline{F(t_i'^{*})}^{\alpha}| < \varepsilon''/(b-a).$$

Hence $|P|<\delta$ implies that for any $\alpha\in[0,1]$

$$d(U(F,P)^{\alpha},L(F,P)^{\alpha}) < \sum_{i=1}^{n} \frac{\max(\varepsilon', \varepsilon'')}{b-a}(t_i - t_{i-1}) = \varepsilon .$$

From this it follows that

$$D(U(F,P),L(F,P)) < \varepsilon . \qquad (\#)$$

It that follows from Theorem 3.2 that F is integrable on [a,b] .

Theorem 4.2. If F is continuous on [a,b] , then corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$D\left(\int_a^b F , \sum_{i=1}^{n} F(\tau_i)\cdot(t_i - t_{i-1})\right) < \varepsilon$$

for each partition P, with mesh $|P| < \delta$ and all $\tau_i \in [t_{i-1} , t_i]$.

Proof. For any partition P and any choise of $\tau_i \in [t_{i-1}, t_i]$

$$L(F,P) \leqslant \sum_{i=1}^{n} F(\tau_i)\cdot(t_i - t_{i-1}) \leqslant U(F,P) .$$

Now, if F is continuous on [a,b] , then by Theorem 4.1 F is integrable on [a,b] and, by Theorem 3.1

$$L(F,P) \leqslant \int_a^b F \leqslant U(F,P).$$

Therefore

$$D\left(\int_a^b F , \sum_{i=1}^{n} F(\tau_i)(t_i - t_{i-1})\right) \leqslant D(U(F,P),L(F,P)).$$

The conclusion of this theorem is then a consequence of (#).

5. Basic properties of the integral of fuzzy mapping.

Lemma 5.1. If F is integrable on [c,d] and $[a,b] \subset [c,d]$, then F is integrable on [a,b] .

Proof. Since F is integrable on [c,d] , we know that corresponding to each $\varepsilon > 0$ there is a partition P of [c,d] such that

$$D(U(F,P),L(F,P)) < \varepsilon .$$

Let P_1 be the partition P plus the division points a,b. Then P_1 is a refinement of P and

$$D(U(F,P_1),L(F,P_1)) < \varepsilon.$$

Let $P_1' = P_1 \cap [a,b]$, i.e. if $t_k, t_{k+1}, \dots, t_l$ are the points of P_1 in $[a,b]$, then $P_1' = \{a=t_k, t_{k+1}, \dots, t_l=b\}$. Now for any $\alpha \in [0,1]$

$$d(U(F,P_1')^\alpha, L(F,P_1')^\alpha) = d\left(\sum_{i=k+1}^{l} U_i(F)^\alpha \cdot (t_i-t_{i-1}), \sum_{i=k+1}^{l} L_i(F)^\alpha \cdot (t_i-t_{i-1})\right) =$$

$$= \max\left(\left|\sum_{i=k+1}^{l} \underline{U_i(F)^\alpha} \cdot (t_i-t_{i-1}) - \sum_{i=k+1}^{l} \underline{L_i(F)^\alpha} \cdot (t_i-t_{i-1})\right|, \left|\sum_{i=k+1}^{l} \overline{U_i(F)^\alpha} \cdot (t_i-t_{i-1}) - \sum_{i=k+1}^{l} \overline{L_i(F)^\alpha} \cdot (t_i-t_{i-1})\right|\right) =$$

$$= \max\left(\left|\sum_{i=k+1}^{l} (\underline{U_i(F)^\alpha} - \underline{L_i(F)^\alpha}) \cdot (t_i-t_{i-1})\right|, \left|\sum_{i=k+1}^{l} (\overline{U_i(F)^\alpha} - \overline{L_i(F)^\alpha}) \cdot (t_i-t_{i-1})\right|\right) \leq$$

$$\leq \max\left(\left|\sum_{i=1}^{n} (\underline{U_i(F)^\alpha} - \underline{L_i(F)^\alpha}) \cdot (t_i-t_{i-1})\right|, \left|\sum_{i=1}^{n} (\overline{U_i(F)^\alpha} - \overline{L_i(F)^\alpha}) \cdot (t_i-t_{i-1})\right|\right) =$$

$$= d(U(F,P_1)^\alpha, L(F,P_1)^\alpha).$$

So,

$$D(U(F,P_1'), L(F,P_1')) \leq D(U(F,P_1), L(F,P_1)) < \varepsilon .$$

Hence F is integrable on $[a,b]$, and this completes the proof.

Lemma 5.2. If F_1 and F_2 are bounded on $[a,b]$, then

1) $$\operatorname{glb} F_1([a,b]) + \operatorname{glb} F_2([a,b]) \leq \operatorname{lub} (F_1+F_2)([a,b]) \leq \operatorname{lub} F_1([a,b]) + \operatorname{lub} F_2([a,b]),$$

$$\operatorname{glb} F_1([a,b]) + \operatorname{glb} F_2([a,b]) \leq \operatorname{glb} (F_1+F_2)([a,b]) \leq \operatorname{lub} F_1([a,b]) + \operatorname{lub} F_2([a,b]),$$

2) $$\operatorname{lub} (\lambda F_1)([a,b]) = \begin{cases} \lambda \cdot \operatorname{lub} F_1([a,b]) & \text{if } \lambda > 0 \\ \lambda \cdot \operatorname{glb} F_1([a,b]) & \text{if } \lambda < 0. \end{cases}$$

From the above Lemma it follow the following properties of the integral of fuzzy mappings:

Theorem 5.1. If F and G are integrable on [a,b], then F+G is integrable on [a,b] and

$$\int_a^b (F+G) = \int_a^b F + \int_a^b G.$$

Theorem 5.2. If F is integrable on [a,b] and λ is a constant different than zero, then $\lambda\cdot F$ is integrable on [a,b] and

$$\int_a^b \lambda\cdot F = \lambda\cdot\int_a^b F .$$

Theorem 5.3. If for any $t \in [a,b]$, $F(t) \geq \overline{0}$, then $\int_a^b F \geq \overline{0}$.

Theorem 5.4. If for any $t \in [a,b]$, $F(t) = C \in L(R)$, then

$$\int_a^b C = C\cdot(b-a).$$

Theorem 5.5. If $c \in [a,b]$ and F is integrable on [a,c] and [c,b], then F is integrable on [a,b] and

$$\int_a^b F = \int_a^c F + \int_c^b F .$$

Theorem 5.6. If F and G are integrable on [a,b] and $F(t) \leq G(t)$ for each $t \in [a,b]$, then

$$\int_a^b F \leq \int_a^b G .$$

References

[1] M. Matłoka, Sequences of fuzzy numbers, BUSEFAL (in print).