LIMITS AND CONTINUITY OF THE FUZZY FUNCTIONS

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0. Introduction. In this paper the limit and continuity of a fuzzy function is presented. A fuzzy function is defined as a mapping from a set of fuzzy numbers to a set of fuzzy numbers. If domain of fuzzy function is a set of real numbers then such fuzzy function we will called the fuzzy mapping.

A first Section is devoted to the limit of a fuzzy function. First the limit of a fuzzy function is defined and next the limit theorems are formulated and proved. In the second Section a continuous fuzzy function is defined and the theorems of continuous fuzzy functions are proved.

1. Limit of a fuzzy function.

Let D and V denote two sets of fuzzy numbers.

Definition 1.1. A fuzzy function F from a set D to a set V is a mapping from D to V.

In the other words, to each fuzzy number $X \in D$ corresponds a fuzzy number $F(X)$ from V.

Definition 1.2. The fuzzy number L is said to be the limit of the fuzzy function F at X_0 if corresponding to each number $\varepsilon > 0$, there is a number $r > 0$, such that whenever $X \in S(X_0,r)$ then $D(F(X),L) < \varepsilon$, where $S(X_0,r) = \{ X : D(X_0,X) < r \text{ and } X \neq X_0 \}$ and D denotes a distance between two fuzzy numbers F(X) and L (see [1]

The notations

$$\lim_{X_0} F = L \quad \text{and} \quad \lim_{X \to X_0} F(X) = L$$

are used to denote that L is the limit of F at X_0.

So,

$$(*) \quad \lim_{X_0} F(X) = L \text{ if and only if } \bigwedge_{\varepsilon > 0} \bigvee_{r > 0} \bigwedge_{X \in S(X_0,r)} D(F(X),L) < \varepsilon$$

Definition 1.3. The fuzzy number L is said to be the limit of the fuzzy function F at X_0 if for any sequence $\{X_n\}$ of fuzzy numbers such that $X_n \neq X_0$

$$(**) \quad \lim_{n \to \infty} X_n = X_0 \Rightarrow \lim_{n \to \infty} F(X_n) = L .$$

Theorem 1.4. The definitions 1.2 and 1.3 of the limit of the fuzzy function are equiponderant.

Proof. Let for any $\varepsilon > 0$ there exists $r > 0$ such that $(*)$ holds and let $\lim_{n \to \infty} X_n = X_0$, $X_n \neq X_0$. Because X_0 is the limit of $\{X_n\}$, so there exists a number N such that for any $n > N$, $X_n \in K(X_0,r)$, (see [1]). From $(*)$ implies that $D(F(X),L) < \varepsilon$. Hence $\lim_{n \to \infty} F(X_n) = L$.

Now, let us assume that $(**)$ holds and for some $\varepsilon = \varepsilon_0 > 0$ does not exist $r > 0$ such that $(*)$ holds. Then for each $r = 1/n$, $n=1,2,\ldots$ there exists a fuzzy number X_n such that $X_n \in K(X_0,1/n)$ and

$X_n \in S(X_0, 1/n)$ and $D(F(X_n), L) \geq \varepsilon_0$. This means that $X_n \neq X_0$, $\lim_{n\to\infty} X_n = X_0$ and L is not the limit of the sequence $\{F(X_n)\}$ in contradiction with (**) . Hence (*) holds. The proof is complete.

Theorem 1.2. If $\lim_{X\to X_0} F(X) = L_1$ and $\lim_{X\to X_0} F(X) = L_2$ then $L_1 = L_2$.

Proof. Let us assume that $\lim X_n = X_0$, $\lim X_n' = X_0$ and $\lim F(X_n) = L_1$, $\lim F(X_n) = L_2 \neq L_1$. Then the sequense $\{Y_n\}$, where $Y_{2n-1} = X_n$, $Y_{2n} = X_n'$ converges to X_0 but the sequence $\{F(Y_n)\}$ diverges - a contradiction.

Definition 1.4. The fuzzy number L is said to be the left-hand (right-hand) limit of the fuzzy function F at X_0 if for any sequence $\{X_n\}$ of fuzzy numbers such that $X_n < X_0$ $(X_n > X_0)$ (see [1])

$$\lim_{n\to\infty} X_n = X_0 \quad \Rightarrow \quad \lim_{n\to\infty} F(X_n) = L .$$

The notations

$$\lim_{X_0^-} F = L \quad \text{and} \quad \lim_{X\to X_0^-} F(X) = L$$

and

$$\lim_{X_0^+} F = L \quad \text{and} \quad \lim_{X\to X_0^+} F(X) = L$$

are used to denote that L is the left-hand or right-hand limit of F at X_0 respectively.

We may give the another but equiponderant definition of the left-hand and right-hand limit of the fuzzy function.

Definition 1.5. The fuzzy number L is said to be the left-hand (right-hand) limit of the fuzzy function F at X_0 if corresponding to each number $\varepsilon > 0$, there is a number $r > 0$ such that whenever

$X \in S^-(X_0, r)$ $(X \in S^+(X_0, r))$ then $D(F(X), L) < \varepsilon$.

So,

$$\lim_{X \to X_0^-} F(X) = L \text{ if and only if } \bigwedge_{\varepsilon > 0} \bigvee_{r > 0} \bigwedge_{X \in S^-(X_0, r)} D(F(X), L) < \varepsilon ,$$

and

$$\lim_{X \to X_0^+} F(X) = L \text{ if and only if } \bigwedge_{\varepsilon > 0} \bigvee_{r > 0} \bigwedge_{X \in S^+(X_0, r)} D(F(X), L) < \varepsilon .$$

Theorem 1.3. The fuzzy number L is the limit of the fuzzy function F at X_0 if and only if there exist the right-hand and left-hand limits of F at X_0 and are equal.

We omit the proof of this Theorem since it is the same as the proof of the corresponding theorem in classical analysis.

Theorem 1.4. If F and G are fuzzy functions and $\lim_{X_0} F = L_1$ and $\lim_{X_0} G = L_2$ then

$$\lim_{X_0} (F + G) , \lim_{X_0} (F - G) , \lim_{X_0} (F \cdot G) , \lim_{X_0} (F/G)$$

exist (for F/G under the assumtions $0 \notin \operatorname{supp} L_2$ and $0 \notin \operatorname{supp} G(X)$ for any X) and

$$\lim_{X_0} (F + G) = \lim_{X_0} F + \lim_{X_0} G = L_1 + L_2$$

$$\lim_{X_0} (F - G) = \lim_{X_0} F - \lim_{X_0} G = L_1 - L_2$$

$$\lim_{X_0} (F \cdot G) = \lim_{X_0} F \cdot \lim_{X_0} G = L_1 \cdot L_2$$

$$\lim_{X_0} (F/G) = \lim_{X_0} F / \lim_{X_0} G = L_1/L_2 .$$

This Theorem implies from the Definition 1.3 and from the correspo-ding theorem for the sequences of fuzzy numbers (see [1]).

Corollary. If $\lim_{X_0} F = L$, then $\lim_{X_0} cF = cL$, where c is a real number.

Theorem 1.5. If $\lim_{X_0} F = L$ and $\lim_{Y_0} G = X_0$ and if there exists a number $r > 0$ such that $G(Y) \neq X_0$ whenever $0 < D(Y,Y_0) < r$, then $\lim_{Y_0} (F \circ G) = L$.

Proof. Since $\lim_{X_0} F = L$, corresponding to any number $\varepsilon > 0$ there is a number $\rho > 0$ such that

$$D(F(X),L) < \varepsilon \qquad (0 < D(X,X_0) < \rho).$$

We may replace this inequality by

$$D(F(G(Y)),L) < \varepsilon \qquad (0 < D(G(Y),X_0) < \rho) \qquad (*)$$

Since $\lim_{Y_0} G = X_0$, there is a number $r_1 > 0$ such that

$$D(G(Y),X_0) < \rho \qquad (0 < D(Y,Y_0) < r_1).$$

Since by hypothesis $D(G(Y),X_0) > 0$ whenever $0 < D(Y,Y_0) < r$, if we let $\bar{r}$ be the smaller of the two numbers r_1 and r, then we have

$$0 < D(G(Y),X_0) < \rho \qquad (0 < D(Y,Y_0) < \bar{r}) \qquad (**)$$

Combining $(*)$ and $(**)$, we have

$$D(F(G(Y)),L) < \varepsilon \qquad (0 < D(Y,Y_0) < \bar{r}).$$

That is,

$$\lim_{Y_0} (F \circ G) = \lim_{Y_0} F(G(Y)) = L.$$

2. Continuity of a fuzzy function.

Definition 2.1. The fuzzy function F is continuous at the fuzzy number X_0 if for each $\varepsilon > 0$ there exists a number $r > 0$ such that

$$D(F(X),F(X_0)) < \varepsilon \qquad \text{whenever} \quad D(X,X_0) < r.$$

We may give the another but equiponderant definition of the continuity of the fuzzy function.

Definition 2.2. The fuzzy function F is continuous at the fuzzy number X_0 if for any sequence $\{X_n\}$ of fuzzy numbers such that $\lim_{n\to\infty} X_n = X_0$ we have $\lim_{n\to\infty} F(X_n) = F(X_0)$.

The above definitions are equivalent to : The fuzzy function F is continuous at the fuzzy number X_0 if $\lim_{X_0} F = F(X_0)$.

Theorem 2.1. If the fuzzy functions F and G are continuous at X_0, then F + G, F - G, and $F\cdot G$ are continuous at X_0, and F/G is continuous at X_0 provided $0 \notin \text{supp}\, G(X_0)$.

Proof. This theorem implies from the Theorem1.4.

Theorem 2.2. If F is continuous at X_0, $\lim_{Y_0} G = X_0$, then $\lim_{Y_0} (F\circ G) = F(X_0)$.

Proof. Since F is continuous at X_0, corresponding to any number $\varepsilon > 0$ there is a number $r > 0$ such that

$$D(F(X),F(X_0)) < \varepsilon \qquad (*)$$

whenever $D(X,X_0) < r$. Also since $\lim_{Y_0} G = X_0$, corresponding to $r > 0$ there is a number $r_1 > 0$ such that

$$D(G(Y),X_0) < r \qquad (**)$$

whenever $0 < D(Y,Y_0) < r_1$. Now, if $0 < D(Y,Y_0) < r_1$ then by (**) $D(G(Y),X_0) < r$. Moreover by (*) $D(F(G(Y)),F(X_0)) < \varepsilon$.
Thus we have shown that corresponding to any number $\varepsilon > 0$ there is a number $r > 0$ such that

$$D((F\circ G)(Y),F(X_0)) < \varepsilon$$

whenever $0 < D(Y,Y_0) < r$.
That is, $\lim_{Y_0} (F\circ G) = F(X_0)$.

Corollary. If G is continuous at Y_0 and F is continuous at $G(Y_0)$, then $F \circ G$ is continuous at Y_0 .

References

[1] Marian Matłoka , Sequences of fuzzy numbers, BUSEFAL (in print).