SEQUENCES OF FUZZY NUMBERS

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1. Introduction. This paper is concerned with sequences of fuzzy numbers. A sequence of fuzzy numbers is defined as a function whose domain is the set of positive integers and whose range is a set of fuzzy numbers. The results which we have received in this paper are similar in forme to the classical theory.
The Section 2 is devoted to notations and terminology. In Section 3 we define the limit of a sequence of fuzzy numbers and next we present the basic properties of the limits. The Section 4 is devoted to algebraic properties of the limits of sequences of fuzzy numbers. Material contained in this paper will be used in the next papers in which we will define the limit of fuzzy function, continuous fuzzy function and integral of fuzzy function.

2. Interval numbers and fuzzy numbers.

By an interval we mean a closed bounded set of "real" numbers

$$[a,b] = \{ x : a \leq x \leq b \} .$$

If A is an interval, we will denote its endpoints by $\underline{A}$ and $\overline{A}$. Thus, $A = [\underline{A}, \overline{A}]$. We will not distinguish between the degenerate interval $[a , a]$ and the real number, a .

We can extend the order relation , $\leq$, on the real line to intervals as follows:

$A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$.

Moreover we define the relation, $<$, in the following way:

$A < B$ if and only if $\underline{A} < \underline{B}$ and $\overline{A} < \overline{B}$.

We can treat intervals A and B as numbers, adding them as follows $A + B = C$ where $\underline{C} = \underline{A} + \underline{B}$ and $\overline{C} = \overline{A} + \overline{B}$.

Put another way, we can add the inequalities

$\underline{A} \leq a \leq \overline{A}$ and $\underline{B} \leq b \leq \overline{B}$

to obtain $\underline{A} + \underline{B} \leq a + b \leq \overline{A} + \overline{B}$. Thus, we can compute the set

$A + B = \{a + b : a \in A, b \in B\}$.

Thus, the sum of two intervals is again an interval.

Similarly, we define the negative of an interval by

$-A = -[\underline{A},\overline{A}] = [-\overline{A},-\underline{A}] = \{-a : a \in A\}$.

For the difference of two intervals, we form

$B - A = B + (-A) = \{b - a : a \in A, b \in B\}$.

More briefly, the rules for interval addition and subtraction are:

$[\underline{A},\overline{A}] + [\underline{B},\overline{B}] = [\underline{A} + \underline{B},\overline{A} + \overline{B}]$

$[\underline{A},\overline{A}] - [\underline{B},\overline{B}] = [\underline{A} - \overline{B},\overline{A} - \underline{B}]$.

We can define the reciprocal of an interval as follows

$1/A = \{1/a : a \in A\}$.

If A is an interval not containing the number 0, then

$1/A = [1/\overline{A}, 1/\underline{A}]$.

For the product of two intervals, we define

$A \cdot B = \{a \cdot b : a \in A, b \in B\}$.

It is not hard to see that $A \cdot B$ is again an interval, whose endpoints can be computed from

$\underline{A \cdot B} = \min(\underline{A}\cdot\underline{B}, \overline{A}\cdot\overline{B}, \underline{A}\cdot\overline{B}, \overline{A}\cdot\underline{B})$

$\overline{A \cdot B} = \max(\underline{A}\cdot\underline{B}, \overline{A}\cdot\overline{B}, \underline{A}\cdot\overline{B}, \overline{A}\cdot\underline{B})$.

In the set of intervals we can define the distance between two intervals A and B as follows:

$$d(A,B) = \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|).$$

All definitions which are presented above we may find in [2] .

Let R be a set of real numbers. Denote

$$L(R) = \{X : R \to [0,1], X \text{ satisfies (i)-(iv) below}\},$$

where

(i) X is normal i.e. there exists an $t_0 \in R$ such that $X(t_0)=1$,

(ii) X is fuzzy convex,

(iii) X is upper semicontinuous,

(iv) $X^0 = \{t \in R : X(t) > 0\}$ is compact .

For $0 < \alpha \leq 1$ denote $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. Then from (i)-(iv) it follows that the α -level set X is an interval for all $0 \leq \alpha \leq 1$.

A function $X : R \to [0,1]$ which satisfies (i) and (ii) is called a fuzzy number. In this paper we will consider only fuzzy numbers from L(R).

We can extend the order relation , $\leq$, as follows:

$X \leq Y$ if and only if for any $\alpha \in [0,1]$ $X^\alpha \leq Y^\alpha$.

Moreover we define the relation , $<$, in the following way:

$X < Y$ if and only if for any $\alpha \in [0,1]$ $X^\alpha < Y^\alpha$.

Define $D : L(R) \times L(R) \to R_+ \cup \{0\}$ by the equation

$$D(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

It is easy to show that D is a metric in L(R). Furthermore, as obtained by Puri and Ralescu [3] , (L(R), D) is a complete metric space.

3. The limit of a sequense of fuzzy numbers.

Definition 3.1. A sequence of fuzzy numbers in R is a function whose domain is the set of positive integers and whose range is a set of fuzzy numbers in R.

Thus, a sequence of fuzzy numbers X is a correspondence from the set of positive integers to a set of fuzzy numbers, i.e., to each positive integer n there corresponds a fuzzy number X(n). It is more common to write X_n rather than X(n) and to denote the sequence by $\{X_n\}$ rather than X. The fuzzy number X_n is called the nth term of the sequence.

An example of a sequence $\{X_n\}$ of fuzzy numbers on R is given by the rule of correspondence

$$X_n(t) = \begin{cases} \dfrac{n}{2n-1}\, t & \text{if} \quad t \in [0, \dfrac{2n-1}{n}), \\ 1 & \text{if} \quad t \in [\dfrac{2n-1}{n}, \dfrac{2n+1}{n}], \\ \dfrac{-n}{2n-1}(t-4) & \text{if} \quad t \in (\dfrac{2n+1}{n}, 4], \\ 0 & \text{otherwise.} \end{cases} \qquad (*)$$

Definition 3.2. The limit of $\{X_n\}$ is X_0, denoted by $\lim X_n = X_0$ or $\lim_{n\to\infty} X_n = X_0$, if for each $\varepsilon > 0$ there exists a number N such that $D(X_n, X_0) < \varepsilon$ whenever n N.

So,

$$\lim_{n\to\infty} X_n = X_0 \quad \text{if and only if} \quad \bigwedge_{\varepsilon > 0} \bigvee_{N} \bigwedge_{n > N} D(X_n, X_0) < \varepsilon .$$

If $\lim X_n$ exists, then we say that the sequence $\{X_n\}$ converges. If a sequence does not converge, then we say that it diverges.

Example. Show that $\lim_{n\to\infty} X_n = X_0$ if X_n is defined by the formula (*) and

$$X_0(t)=\begin{cases}\frac{1}{2}t & \text{if } t\in[0,2),\\ -\frac{1}{2}(t-4) & \text{if } t\in[2,4],\\ 0 & \text{otherwise.}\end{cases}$$

Solution. Take $\varepsilon > 0$. We wish to show that there exists a number N such that $D(X_n, X_0) < \varepsilon$ whenever $n > N$. But $D(X_n, X_0) = \frac{1}{n}$. So, we must show that there exists a number N such that $D(X_n, X_0) = \frac{1}{n} < \varepsilon$ whenever $n > N$. If we let $N = \frac{1}{\varepsilon}$, then $n > N$ implies that $\frac{1}{n} < \frac{1}{N} = \varepsilon$. Thus, we have shown that $\lim_{n\to\infty} X_n = X_0$.

Definition 3.2. A neighborhood of fuzzy number X_0 of radius ε denoted by $K(X_0, \varepsilon)$, is the set of all fuzzy numbers X from $L(R)$ such that $D(X_0, X) < \varepsilon$.

Theorem 3.1. A fuzzy number X_0 is a limit of a sequence $\{X_n\}$ if and only if every neighborhood of X_0 contains infinitely many terms of the sequence.

Proof. Let X_0 is a limit of $\{X_n\}$. This means, that

$$\bigwedge_{\varepsilon>0}\ \bigvee_{N}\ \bigwedge_{n>N} D(X_n,X_0)<\varepsilon \quad \text{iff} \quad \bigwedge_{\varepsilon>0}\ \bigvee_{N}\ \bigwedge_{n>N} X_n\in K(X_0,\varepsilon).$$

Theorem 3.2. If a sequence $\{X_n\}$ is converged then it has only one limit.

Proof. Assume $\lim X_n = X_0$. Let Y_0 be any fuzzy number distinct from X_0. Take a neighborhood K of X_0 and a neighborhood K' of Y_0 such that $K \cap K' = 0$. Since $\lim X_n = X_0$, all but a finite number of terms of $\{X_n\}$ lie in K. Therefore, the neighborhood K' of Y_0 cannot contain infinitely many terms of $\{X_n\}$. This shows that Y_0 cannot be a limit of $\{X_n\}$ and, hence, if $\lim X_n = X_0$, $\{X_n\}$ can have only the one limit X_0 .

Theorem 3.3 If there exists n_0 such that for all $n > n_0$ $X_n \leq Y_n \leq Z_n$ and if $\lim X_n = B = \lim Z_n$, then $\{Y_n\}$ converges and $\lim Y_n = B$.

Proof. Take $\varepsilon > 0$. Since $\lim X_n = B$, there exists a number n_1 such that

$$D(X_n, B) < \varepsilon \qquad \text{whenever} \quad n > n_1 .$$

Since $\lim Z_n = B$, there exists a number n_2 such that

$$D(Z_n, B) < \varepsilon \qquad \text{whenever} \quad n > n_2 .$$

Let $\bar{n} = \max (n_0, n_1, n_2)$. Then

$$D(Y_n, B) \leq D(Y_n, Z_n) + D(Z_n, B) \leq D(X_n, Z_n) + D(Z_n, B) \leq$$

$$\leq D(X_n, B) + D(Z_n, B) + D(Z_n, B) < 3\varepsilon$$

$$\text{whenever} \quad n > \bar{n} ,$$

and therefore $\lim Y_n = B$.

Definition 3.3. A sequence $\{X_n\}$ is called bounded if the set of fuzzy numbers $\{X_n : n = 1, 2, \dots\}$ is bounded, i.e., there exist two fuzzy numbers L and U such that for any n

$$L \leq X_n \leq U .$$

The least upper bound of a set $\{D(X, Y) : X, Y \in \mathcal{A}\}$ is called the diameter of $\mathcal{A}$, written $\delta(\mathcal{A})$.

Let us note that if $\delta(\mathcal{A})$ is finite then $\mathcal{A}$ is bounded.

Theorem 3.4. Any converged sequence $\{X_n\}$ of fuzzy numbers is bounded.

Proof. Let $X_0 = \lim X_n$ and let $\mathcal{A} = \{X_n : n = 1, 2, \dots\}$. From the Definition 3.2 there exists a number N such that $D(X_n, X_0) < 1$ whenever $n > N$. Let $\delta = \max (D(X_1, X_0), D(X_2, X_0), \dots,$

$D(X_N, X_0), 1)$. Then for any n $D(X_n, X) \le \delta$. But D is a metric in L(R). So,

$$D(X_n, X_m) \le D(X_n, X_0) + D(X_0, X_m) \le 2\delta .$$

This means that $\delta(\mathcal{A}) \le 2\delta$ i.e. the sequence $\{X_n\}$ is bounded.

Theorem 3.5. If for any n $X_n = X_0$ then $\lim X_n = X_0$.

Definition 3.4. If $\{X_n\}$ is a sequence of fuzzy numbers and $\{n_k\}$ is an increasing sequence of positive integers, then the sequence $\{X_{n_k}\}$ is called a subsequence of $\{X_n\}$.

Theorem 3.6. If $\{X_n\}$ converges, then any subsequence of $\{X_n\}$ converges to the same point.

Proof. If $\lim X_n = X_0$, we wish show that, for any subsequence $\{X_{n_k}\}$ of $\{X_n\}$, $\lim X_{n_k} = X_0$. Thus, we wish to show, for any $\varepsilon > 0$, there exists a number k_0 such that $X_{n_k} \in K(X_0, \varepsilon)$ for all $k > k_0$. Take $\varepsilon > 0$. Since $\lim X_n = X_0$, there exists a number N such that $X_n \in K(X_0, \varepsilon)$ for all $n > N$. Also, since $\{n_k\}$ is an increasing sequence of positive integers there exists a number k_0 such that $n_k > N$ whenever $k > k_0$. Thus, if $k > k_0$, $n_k > N$ and hence $X_{n_k} \in K(X_0, \varepsilon)$.

4. Algebraic properties of the limits of sequences of fuzzy numbers.

Theorem 4.1. If $\lim X_n = X_0$ and $\lim Y_n = Y_0$, then

(1) $\lim (X_n + Y_n) = X_0 + Y_0$,

(2) $\lim (X_n - Y_n) = X_0 - Y_0$,

(3) $\lim (X_n \cdot Y_n) = X_0 \cdot Y_0$,

(4) $\lim X_n/Y_n = X_0/Y_0$ if for all n $0 \notin \operatorname{supp} Y_n$ and $0 \notin \operatorname{supp} Y_0$.

For the operations on fuzzy numbers see e.g. [1] .

Proof. (1) Let $\lim X_n = X_0$ and $\lim Y_n = Y_0$. Then for any $\varepsilon > 0$ there exists a number N such that

$$D(X_n, X_0) < \varepsilon \quad \text{and} \quad D(Y_n, Y_0) < \varepsilon \qquad \text{whenever} \quad n > N.$$

From the definition of D it follows that for any $\alpha \in [0,1]$

$$d(X_n^\alpha, X_0^\alpha) = \max (|\underline{X}_n^\alpha - \underline{X}_0^\alpha| , |\overline{X}_n^\alpha - \overline{X}_0^\alpha|)$$

and

$$d(Y_n^\alpha, Y_0^\alpha) = \max (|\underline{Y}_n^\alpha - \underline{Y}_0^\alpha| , |\overline{Y}_n^\alpha - \overline{Y}_0^\alpha|) .$$

Consider

$$d(X_n^\alpha + Y_n^\alpha , X_0^\alpha + Y_0^\alpha) = d([\underline{X}_n^\alpha + \underline{Y}_n^\alpha , \overline{X}_n^\alpha + \overline{Y}_n^\alpha],[\underline{X}_0^\alpha + \underline{Y}_0^\alpha, \overline{X}_0^\alpha + \overline{Y}_0^\alpha])$$

$$= \max (|\underline{X}_n^\alpha + \underline{Y}_n^\alpha - \underline{X}_0^\alpha - \underline{Y}_0^\alpha| , |\overline{X}_n^\alpha + \overline{Y}_n^\alpha - \overline{X}_0^\alpha - \overline{Y}_0^\alpha|) \leq$$

$$\leq \max (|\underline{X}_n^\alpha - \underline{X}_0^\alpha| + |\underline{Y}_n^\alpha - \underline{Y}_0^\alpha|, |\overline{X}_n^\alpha - \overline{X}_0^\alpha| + |\overline{Y}_n^\alpha - \overline{Y}_0^\alpha|) \leq$$

$$\leq \max (|\underline{X}_n^\alpha - \underline{X}_0^\alpha| , |\overline{X}_n^\alpha - \overline{X}_0^\alpha|) + \max (|\underline{Y}_n^\alpha - \underline{Y}_0^\alpha|, |\overline{Y}_n^\alpha - \overline{Y}_0^\alpha|) =$$

$$= d(X_n^\alpha, X_0^\alpha) + d(Y_n^\alpha, Y_0^\alpha) < 2\varepsilon .$$

So, for any $\alpha \in [0,1]$ $\quad d(X_n^\alpha + Y_n^\alpha , X_0^\alpha + Y_0^\alpha) < 2\varepsilon$.

Hence,

$$D(X_n + Y_n, X_0 + Y_0) < 2\varepsilon$$

and this completes the proof of (1).

For (2) the proof is analogous to (1).

(3) Now, let us consider $d(X_n^\alpha \cdot Y_n^\alpha, X_0^\alpha \cdot Y_0^\alpha)$. From the Section 2 it follows that the endpoints of $X_n^\alpha \cdot Y_n^\alpha$ and $X_0^\alpha \cdot Y_0^\alpha$ depend on the signs of the endpoints of X_n^α , Y_n^α and X_0^α , Y_0^α respectively. We will prove (3) only for the following situation: $\underline{X}_n^\alpha , \underline{Y}_n^\alpha , \underline{X}_0^\alpha , \underline{Y}_0^\alpha > 0$. The proofs of the other situations are similar. So,

$$d(X_n^\alpha \cdot Y_n^\alpha, X_0^\alpha \cdot Y_0^\alpha) = d([\underline{X}_n^\alpha \underline{Y}_n^\alpha , \overline{X}_n^\alpha \overline{Y}_n^\alpha] , [\underline{X}_0^\alpha \underline{Y}_0^\alpha , \overline{X}_0^\alpha \overline{Y}_0^\alpha]) =$$

$$= \max (|\underline{X}_n^\alpha \underline{Y}_n^\alpha - \underline{X}_0^\alpha \underline{Y}_0^\alpha| , |\overline{X}_n^\alpha \overline{Y}_n^\alpha - \overline{X}_0^\alpha \overline{Y}_0^\alpha|) \leq$$

$$\leq \max (|\underline{X}_n^\alpha - \underline{X}_0^\alpha| \cdot |\underline{Y}_n^\alpha| + |\underline{Y}_n^\alpha - \underline{Y}_0^\alpha| \cdot |\underline{X}_0^\alpha| ,$$

$$|\overline{X}_n^\alpha - \overline{X}_0^\alpha| \cdot |\overline{Y}_n^\alpha| + |\overline{Y}_n^\alpha - \overline{Y}_0^\alpha| \cdot |\overline{X}_0^\alpha|) .$$

Because $\{X_n\}$ and $\{Y_n\}$ converge, so they are bounded. From this it follows that there exists a number k_0 such that $|\underline{Y}_n^\alpha|$, $|\overline{Y}_n^\alpha|$, $|\underline{X}_n^\alpha|$ and $|\overline{X}_n^\alpha|$ are less than k_0.

Hence

$$\begin{aligned} d(X_n^\alpha \cdot Y_n^\alpha, X_0^\alpha \cdot Y_0^\alpha) \leq \; & \max(|\underline{X}_n^\alpha - \underline{X}_0^\alpha| \cdot k_0 + |\underline{Y}_n^\alpha - \underline{Y}_0^\alpha| \cdot k_0, \\ & |\overline{X}_n^\alpha - \overline{X}_0^\alpha| \cdot k_0 + |\overline{Y}_n^\alpha - \overline{Y}_0^\alpha| \cdot k_0) \leq \\ & \leq k_0 \cdot (\max(|\underline{X}_n^\alpha - \underline{X}_0^\alpha|, |\overline{X}_n^\alpha - \overline{X}_0^\alpha|) + \\ & + \max(|\underline{Y}_n^\alpha - \underline{Y}_0^\alpha|, |\overline{Y}_n^\alpha - \overline{Y}_0^\alpha|)) \leq \\ & \leq k_0 \cdot (d(X_n^\alpha, X_0^\alpha) + d(Y_n^\alpha, Y_0^\alpha)) \leq k_0 \cdot 2 \cdot \varepsilon . \end{aligned}$$

Taking $\varepsilon = \varepsilon'/2k_0$ we have

$$D(X_n \cdot Y_n, X_0 \cdot Y_0) < \varepsilon' .$$

So, the proof of (3) is complete.

(4) Let $\lim Y_n = Y_0$. First we show that for any $\alpha \in [0,1]$ $d(1/Y_n^\alpha, 1/Y_0^\alpha) < \varepsilon$. Analogously as in the above proof let us assume that $\underline{Y}_n^\alpha, \overline{Y}_n^\alpha, \underline{Y}_0^\alpha, \overline{Y}_0^\alpha > 0$. Then we have

$$\begin{aligned} d(1/Y_n^\alpha, 1/Y_0^\alpha) & = d([1/\overline{Y}_n^\alpha, 1/\underline{Y}_n^\alpha], [1/\overline{Y}_0^\alpha, 1/\underline{Y}_0^\alpha]) = \\ & = \max(|1/\overline{Y}_n^\alpha - 1/\overline{Y}_0^\alpha|, |1/\underline{Y}_n^\alpha - 1/\underline{Y}_0^\alpha|) = \\ & = \max(|(\overline{Y}_0^\alpha - \overline{Y}_n^\alpha)/\overline{Y}_n^\alpha \cdot \overline{Y}_0^\alpha|, |(\underline{Y}_0^\alpha - \underline{Y}_n^\alpha)/\underline{Y}_n^\alpha \cdot \underline{Y}_0^\alpha|). \end{aligned}$$

Because $\lim Y_n = Y_0$ so we have

$$\begin{aligned} d(1/Y_n^\alpha, 1/Y_0^\alpha) & < \max(|\overline{Y}_n^\alpha - \overline{Y}_0^\alpha|, |\underline{Y}_n^\alpha - \underline{Y}_0^\alpha|) \cdot k_0 = \\ & = d(Y_n^\alpha, Y_0^\alpha) \cdot k_0 < \varepsilon \cdot k_0 . \end{aligned}$$

Taking $\varepsilon = \varepsilon'/k_0$ we have

$$D(1/Y_n, 1/Y_0) < \varepsilon .$$

Hence

$$\lim 1/Y_n = 1/Y_0 .$$

Now, from (3) it follows that

$$\lim X_n/Y_n = \lim X_n \cdot 1/Y_n = X_0 \cdot 1/Y_0 = X_0/Y_0 .$$

References

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