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FUZZY BAGS

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ABSTRACT

Based upon R R. Yager's theory of bags[1], a new definition of fuzzy bags and correspondent operations are given in this paper. Relational bases and example of set-valued statistics are also discussed.

INTRODUCTION

In classical set theory, the elements of a set are different ones in essence. In other words, only one is chosen and repeated elements are redundant in a set. However, in some situations, we may want a structure which is a collection of objects in the same sense as a set but in which redundancy counts. For example, while we search in a name-age-position relational base, number of certain age is always meaningful for us. In set-valued statistics, as we look for the estimate function of membership grade $\mu_{\xi}(x)$, both the value of membership grade of x in interval [0,1] and times of x appears are taken into account.

Bags defined by R R. Yager is in fact a set A composed of same or different elements and characterized by a counting function C_A such that

$$C_A : X \longrightarrow N,$$

where N is the set of non-negative integers. This set A is named the bag and $C_A(x)$ indicates the number of times the element x appears in the bag A. Bag A can also be expressed as

$$A = \langle C_A(x)/x \rangle .$$

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In fuzzy sets, the membership grade of each element should be taken into account. When we extend the definition of fuzzy sets and the concept of crisp bags to the definition of fuzzy bags, we should also consider the times of some element with certain membership degree appears in a set. If X denotes the universe of discourse, U represents the membership grade , C indicates the count of each element, then a tri-dimentional coordinate system forms. Fuzzy bag may be represented by a mapping from X x U to C.

<u>Def.</u> Assume X is the universe of discourse, U is the membership grade, fuzzy bag A is a mapping from Descartes' product X x U to count C characterized by:

$$FC_A(u/x): X \times U \longrightarrow C.$$

where C is non-negative set. Fuzzy bag A can also be denoted as

$$A(u/x)= \langle FC_A(u/x)/(u/x) \rangle .$$

where $FC_A(u/x)$ denotes the count of (u/x), $FC_A(u/x)/(u/x)$ denotes the element of fuzzy bag A.

Fuzzy bag with only one element in it is named the bag of singleton.

<u>Example:</u> An example of two fuzzy bags is

$$A = \langle 2/(0.2/x1),\ 3/(0.3/x1),\ 4/(0.1/x2),\ 5/(0.2/x2) \rangle .$$

$$B = \langle 5/(0.4/x1) \rangle .$$

then $FC_A(0.3/x1)=3$, $FC_A(0.2/x2)=5$, $FC_B(0.4/x1)=5$ and fuzzy bag B is a bag of singleton.

If we consider bags composed of only one $x \in X$, then fuzzy bag will become a crisp bag on unit interval which we called the point-bag, denoted by

$$A(x) = \langle FC_A(u)/u \rangle \ (x) .$$

For above example, the point-bag of A to x1 is

$$A(x1) = \langle 2/0.2 \ , \ 3/0.3 \rangle \ (x1)$$

In fuzzy sets theory, we often assume that for each $x \in X$, the membership degree of x to fuzzy set A is unique. In fuzzy bag, this restriction does not remain. However, if we want to remain this, a special fuzzy bag is formed.

Def. If for fuzzy bag on $X \times U$ and each $x \in X$, membership grade u is unique, we call it canonical fuzzy bag.

Theorem: Let A be fuzzy bag on $X \times U$, if for each $x \in X$, point-bag $A(x)$ is a bag of singleton, then A is a canonical fuzzy bag.

Proof: From the condition of this theorem we have: for each $x \in X$, point-bag $A(x) = \langle FC_A(u/x)/(u/x)\rangle (x)$ is a bag of singleton, in other words, there is only one element in it, which is $FC_A(u/x)/(u/x)$, i.e. for this x , u is unique, hence A is a canonical fuzzy bag. ||

Example: $A(u/x) = \langle 3/(0.2/x1),\ 5/(0.3/x2),\ 4/(0.2/x3)\rangle$ is a canonical fuzzy bag, since

$A(x1)=\langle 3/0.2\rangle$, $A(x2)=\langle 5/0.3\rangle$, $A(x3)=\langle 4/0.2\rangle$, they are all bags of singleton. Whereas the bag A in above example is not the canonical fuzzy bag.

Analogous to fuzzy set, in canonical fuzzy bag, each element x has a unique membership grade, but count is not unique.

Def. Assume A is a fuzzy bag drawn from universe X, the cardinality of A, denoted Card(A) is defined as

$$\mathrm{Card}(A) = \sum_{x \in X} \sum_{u \in U} u \times FC_A(u/x)$$

where x means common product operation.

Def. Assume A is a fuzzy bag drawn from universe X, the absolute cardinality of A, denoted /Card(A)/ is defined as

$$/\mathrm{Card}(A)/ = \sum_{x \in X} \sum_{u \in U} FC_A(u/x).$$

The difference between above two definitions is that the membership grade is considered in the former and not considered in the latter.

Since the point-bag has become the crisp bag upon interval [0,1], the cardinality is as follows:

$$\mathrm{Card}(A(x)) = \sum_{u \in U} u \times FC_A(u/x).$$

$$/\mathrm{Card}(A(x))/ = \sum_{u \in U} FC_A(u/x)\ .$$

From the definition of point-bag, we see that the fuzzy bag is composed of point-bags of their elements, so for operation of cardinality , we have

$$Card(A) = \sum_{x \in X} Card(A(x)) .$$

$$/Card(A)/ = \sum_{x \in X} /Card(A(x))/ .$$

Example: Let A be

$$A = \langle 3/(0.1/x1),\ 2/(0.2/x1),\ 4/(0.3/x2),\ 2/(0.2/x3) \rangle .$$

We have

$$Card(A) = 3x0.1+2x0.2+4x0.3+2x0.2 = 2.3$$

$$/Card(A)/ = 3+2+4+2 = 11$$

Where

$$Card(A(x1)) = 3x0.1+2x0.2 = 0.7$$

$$Card(A(x2)) = 4x0.3 = 1.2$$

$$Card(A(x3)) = 2x0.2 = 0.4.$$

It can be easily verified that

$$Card(A) = Card(A(x1)) + Card(A(x2)) + Card(A(x3)).$$

Formula for absolute cardinality can also be verified .

Crisp bag takes the common subset as its base set or support set, fuzzy bag should also take the fuzzy subset as its support set. Because of the multivaluedness of membership grade in fuzzy bags, it is inconvenient to discuss the support set generally. For this reason, we confine our discussion merely in canonical fuzzy bags.

Def. Let A be canonical fuzzy bag on universe X, B be fuzzy subset of X, if for each $x \in X$ such that

$$FC_B(u/x) = 1, \text{ if } FC_A(u/x) > 0,$$

$$FC_B(u/x) = 0, \text{ if } FC_B(u/x) = 0.$$

then B is named the support set of canonical fuzzy bag A, denoted $\underline{A}$. $\underline{A}$ may be characterized by following formula:

$$FC_{\underline{A}}(u/x) = sgn(FC_A(u/x)).$$

where sgn is the sign function such that

$$sgn(x) = \begin{cases} 1 , & \text{if } x > 0, \\ 0 , & \text{if } x = 0, \\ -1 , & \text{if } x < 0. \end{cases}$$

In general, common fuzzy bags can be resolved into some canonical fuzzy bags, hence their support set can be discussed respectively.

Theorem: If the count of each element in a canonical fuzzy bag equals to 1, then it is a support fuzzy set itself.

Proof: Let A be a canonical fuzzy bag and for each $x \in X$, $FC_A(u/x) \in \{0,1\}$, such that

$$FC_A(u/x) = 1, \text{ if } FC_A(u/x) > 0,$$
$$FC_A(u/x) = 0, \text{ if } FC_A(u/x) = 0.$$

It satisfies the definition of support fuzzy set. ||

To investigate the maximal count of a fuzzy bag, now we introduce the concept of peak-value.

Def. Let A be a fuzzy bag on universe X, the maximal count of elements in A, denoted PV(A), is called the peak-value, such that

$$PV(A) = \max_{\substack{x \in X \\ u \in U}} [\ FC_A(u/x)\ ].$$

For point-bags, peak-value should be

$$PV(A) = \max_{u \in U} [\ FC_A(u/x)\ ].$$

To represent the grade of concentration of counts in a fuzzy bag, now we give the following definition.

Def. The grade of concentation of a fuzzy bag on universe X, denoted S(A), such that

$$S(A) = \frac{PV(A)}{M - m}$$

where $M = \sup_{u \in U} u$, $m = \inf_{u \in U} u$, PV(A) is the peak-value of bag A.

Now let us look at some operations of fuzzy bags.

Def. Let A and B be fuzzy bags on universe X, the addition of these two bags results in a new bag C, denoted $C = A \oplus B$ such that for each $x \in X, u \in U$

$$FC_C(u/x) = FC_A(u/x) + FC_B(u/x).$$

Def. Let A and B be fuzzy bags drawn from universe X, the removal of bag B from A results in a new bag D, denoted $D = A \ominus B$, such that for each $x \in X$, $u \in U$,

$$FC_D(u/x) = \mathrm{Max}[\ FC_A(u/x) - FC_B(u/x), 0].$$

<u>Def.</u> Let A and B be fuzzy bags drawn from universe X, the union of two bags results in a new bag C, denoted $C = A \cup B$ such that for each $x \in X, u \in U$

$$FC_C(u/x) = \mathrm{Max}[\ FC_A(u/x),\ FC_B(u/x)\ ].$$

<u>Def.</u> Let A and B be fuzzy bags drawn from universe X, the intersection of these two bags denoted $D = A \cap B$ such that for each $x \in X$, $u \in U$

$$FC_D(u/x) = \mathrm{Min}[\ FC_A(u/x),\ FC_B(u/x)\ ].$$

<u>Def.</u> Let A and B be two fuzzy bags drawn from universe X, the elements in bag A which also belong to bag B compose a new bag D, denoted $D = A \otimes B$, such that for each $x \in X$, $u \in U$

$$FC_D(u/x) = FC_A(u/x) \times \mathrm{sgn}(FC_B(u/x)).$$

where x is common product.

Note: (1) While B becomes a common fuzzy subset, $FC_B(u/x) \in \{0,1\}$, the expression of definition can be written as

$$FC_D(u/x) = FC_A(u/x) \times FC_B(u/x).$$

(2) While A becomes a crisp bag, the expression of the definition can be written as

$$FC_D(x) = FC_A(x) \times \mathrm{sgn}\,(\sup_{u \in U}[FC_B(u/x)]).$$

Let us look at some compound operations.

$$E = A \otimes (B_1 \cup B_2)$$

From the definition concerned we have

$$FC_E(u/x) = FC_A(u/x) \times \mathrm{sgn}(FC_{B1}(u/x) \vee FC_{B2}(u/x)).$$

$$= [FC_A(u/x) \times \mathrm{sgn}(FC_{B1}(u/x))] \vee [FC_A(u/x) \times \mathrm{sgn}(FC_{B2}(u/x))].$$

Note that the non-negativeness of count function FC, the distribution operation is available, hence

$$E = A \otimes (B_1 \cup B_2) = (A \otimes B_1) \cup (A \otimes B_2).$$

Analogously, we may have

$$G = (A_1 \cap A_2) \circledast B = (A_1 \circledast B) \cap (A_2 \circledast B)$$

$$H = A \circledast (B_1 \cap B_2) = (A \circledast B_1) \cap (A \circledast B_2)$$

$$Q = (A_1 \vee A_2) \circledast B = (A_1 \circledast B) \vee (A_2 \circledast B).$$

APPLICATIONS

Ex.1. According to the method of degree analysis set forth in [2], let us measure the grade of satisfactory psychologically for some design index X1, X2, X3, X4, X5. Here we use the method of line segment, " a " means the point estimation,"m" means the blindness. After experiment we get the following table:

NAME / INDEX	X1		X2		X3		X4		X5	
Item	a	m	a	m	a	m	a	m	a	m
W	.3,	.2	.8,	.2	.15,	.1	.3,	.2	.45,	.5
L	.45,	.2	.85,	.1	.25,	.1	.35,	.1	.5,	.2
Z	.35,	.3	.75,	.1	.4,	.2	.25,	.3	.75,	.1
H	.2,	.2	.75,	.3	.25,	.3	.25,	.3	.45,	.3
C	.65,	.1	.6,	.4	.35,	.5	.3 ,	.4	.45,	.5

Above table is in fact a relational database, dealing with the point estimation only, we get the point-bags as follows:

$A_a(X1) = \langle 1/.3,\ 1/.45,\ 1/.35,\ 1/.2,\ 1/.65 \rangle$

$A_a(X2) = \langle 1/.8,\ 1/.85,\ 2/.75,\ 1/.6 \rangle$

$A_a(X3) = \langle 1/.15,\ 2/.25,\ 1/.35,\ 1/.4 \rangle$

$A_a(X4) = \langle 2/.3,\ 1/.35,\ 2/.25 \rangle$

$A_a(X5) = \langle 3/.45,\ 1/.5,\ 1/.75 \rangle$.

From the formula

a(A(Xi)) = Card(A(Xi))//Card(A(Xi))/

we obtain each point estimation

a(A(X1)) =1.95/5 = .38, a(A(x2)) = 3.75/5 = .75,

a(A(X3)) =1.4/5 = .28, a(A(X4)) = 1.45/5 = .29,

a(A(X5)) =2.6/5 = .52.

Hence we get the point estimation vector

A = (.39, .75, .28, .29, .52).

Likely, we get the blindness vector

M = (.22, .22, .24, .26, .32).

To reflect the concentration of opinions, using the formula of concentration grade

S(A(Xi)) = PV(A(Xi))/(M-m)

where $M = \sup_{u \in U} u$, $m = \inf_{u \in U} u$, PV represents the peak-value, we get

S(A(X1)) = 1/.35 = 2.87, S(A(X2)) = 2/.25 = 8,

S(A(X3)) = 8, S(A(X4)) = 20, S(A(X5)) = 10.

Through normalizing, we get the vector of concentration grade

S = (.05, .16, .16, .5, .25).

In above vectors, A represents the point estimation which implies the average tendency of opinions; M represents the grade of psychological uncertainty of experimenters; S indicates the concentration grade of opionions. According to the principle of greatest membership[4], X2 is considered the most satisfactory index, X5 is the most uncertain index and about X4, the opinions are most concentrating.

Consider X2 only, it is the most satisfactory and most certain index, the opinions about X2 are concentrating either.

<u>Ex2.</u> Preparing an important strategy, it is necessary to take the participants' confidence into account. Let us divide the grade of confidence into a fuzzy vector C = (.1, .25, .5, .75, 1), and make psychological test to ten senior officers. A fuzzy bag is obtained, since consider the confidence only, it is in fact a point-bag, denoted C.

$C = \langle 1/.1,\ 1/.25,\ 3/.5,\ 3/.75,\ 1/1 \rangle.$

Hence, the point estimation of confidence is

A = Card(A)//Card(A)/ = 5.1/10 = .51

To be more secure, it is required that the number of people with confidence grade not less than 0.5 must be greater than half of the total number, forthermore, the cardinality of total confidence must be greater than half of the absolute cardinality. To meet this requirement, first we should constitute a support set:

$B = \langle .5,\ .75,\ 1 \rangle$, then

So, $D = A \ast B = \langle 3/.5,\ 3/.75,\ 1/1 \rangle$.

/Card(D)/ = 7 > 5, Card(A) = 5.1 > 5 = ½/Card(A)/.

In addition to investigate the confidence of senior officers, to guarantee the victory of action, we make more test in ten middle class officers and get another bag E.

$E = \langle 1/.1,\ 2/.25,\ 1/.5,\ 4/.75,\ 2/1 \rangle$.

Connect E with C, characterized by

$H = C \uplus E = \langle 1/.1,\ 2/.25,\ 3/1.5,\ 4/.75,\ 2/1 \rangle$.

It is in fact the higher count synthesis, operating with support set B, we get

$H \circledast B = \langle 3/.5,\ 4/.75,\ 2/1 \rangle$.

$/Card(H \circledast B)/ = 9 > \frac{1}{2}\ /Card(H)/ = 6$

To make the lower count synthesis, we get

$G = C \odot E = \langle 1/.1,\ 1/.25,\ 1/.5,\ 3/.75,\ 1/1 \rangle$.

Hence $G \circledast B = \langle 1/.5,\ 3/.75,\ 1/1 \rangle$, furthermore we get

$/Card(G \circledast B)/ = 5 > \frac{1}{2}\ /Card(G)/ = 3.5$.

To make the addition synthesis, we have

$F = C \oplus E = \langle 2/.1,\ 3/.25,\ 4/.5,\ 7/.75,\ 3/1 \rangle$.

$F \circledast B = \langle 4/.5,\ 7/.75,\ 3/1 \rangle$.

$/Card(F \circledast B)/ = 14 > \frac{1}{2}/Card(F)/ = 9.5$

Above three syntheses represent critical, conservative and average estimations respectively.

CONCLUSION

A new definition of fuzzy bag is set up and some examples are given in this paper. Though the applications are preliminary for the time being, it shows a prospective future.

About the lattice structure of bags, about the bags of type II, corresponding to point-bag, could we set up the membership-bag, about the further applications of bag theory in relational database and in set-valued statistics, there are many avenues of development open in this field.

KEYWORDS

Theory of Bags, Fuzzy Bags, Fuzzy Set Theory

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