SUBGROUPS AND INVARIANT SUBGROUPS OF NORMAL HYPERGROUPS

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This paper is a continuation of the paper [2], the structure of normal hypergroups is discussed here, and their subgroups and invariand subgroups are also obtained.

KEYWORDS: Hypergroup, Normal hypergroup, Invariant Subgroup, Subgroup, Group.

1. INTRODUCTION

The theory of set value mappings need to promote all kinds of mathematical structures from their universes to their power sets with researches in the theoretical basis of Fuzzy mathematics. In Ref.[2] normal hypergroups were reseached and a main consequence was obtained: every normal hypergroup must be generalized quotient group.

What is called a hypergroup is a group formed by that the operation in the group is induced into its power set, a normal hypergroup is a hypergroup whose unit element contains the unit element of group.

Let $(G,\cdot)$ be a group, for any nonempty SubSet E of G, write

$$N(E)=\left\{x\in G \,\middle|\, xE=Ex\right\}$$

it is called a normalizer of G, and it is a subgroup of G.

Let E be a SubSemigroup of G, which contains the unit element of G, then $E^2 = E$.

For any subgroup G_0 of $N(E)$, writ

$$G_0\big|_E = \left\{ aE \;\middle|\; a \in G_0 \right\}$$

We induce an operation on the $G_0\big|_E$: for any aE, bE $\in G_0\big|_E$ (a, b $\in G_0$

$$(aE)\,(bE) = abE^2 = abE$$

so the $G_0\big|_E$ is a group with resperct to the above operation. In Ref.(1) it is called a generalized quotient group which G_0 is with respect to E. In Ref.(2) we have proved that $G_0\big|_E$ also is a normal hypergroup with unit element E. The $G_0\big|_E$ is called normal hypergroup of E after. The $N(E)\big|_E$ is a greatest normal hypergroup of E clearly. In special, when the E is a subgroup of G, this $N(E)\big|_E$ is a quotient group, which N(E) is with respect to E. If the E is a invariant subgroup of G, then N(E) = G, and the $N(E)\big|_E$ is a quotient group, which G is with respect to E. Thus it can be seen that quotient groups are a special case of normal hypergroups. Conversely, the normal hypergroups are an extension of quotient groups.

2. SUBGROUPS AND INVARIANT SUBGROUPS OF NORMAL HYPERGROUPS

Let (G,·) be a group, the subset E containing nuit element of G be a subsemigroup of G.

Now write

$$\bar{E} = \left\{ a \;\middle|\; a \in E,\ a^{-1} \in E \right\}$$

Let e be the unit element of G, from $e \in E$ there is $\bar{E} \neq \phi$, It is called kernel of E. When E is a subsemigroup of G and $e \notin E$, we have $\bar{E} = \phi$.

LEMMA 1. Let E be a SubSemigroup of G and $e \in E$, then

(1). $\bar{E} < G$

(2). $\bar{E} \triangleleft N(E)$

PROOF. (1) For any a, b $\in \bar{E}$, i.e, a, b $\in E$ and $a^{-1}, b^{-1} \in E$.

Sinse E is a SubSemigroup of G, we have $ab^{-1} \in E$, $(ab^{-1})^{-1} = ba^{-1} \in E$, so $ab^{-1} \in \bar{E}$

Therefore $\bar{E}$ is a subgroup of G.

(2). From (1) we know that $\bar{E}$ is a subgroup of N(E).

Now we prove $a\bar{E} = \bar{E}a$ for any $a \in N(E)$.

We assume $X \in \bar{E}$, then $X \in E$ and $X^{-1} \in E$.

From $aE = Ea$ there is $y \in E$ such that $aX = ya$, i.e. $y = aXa^{-1}$, so $y^{-1} = ax^{-1}a^{-1}$, since $X^{-1} \in E$, so $y^{-1} \in E$, i.e. $y \in \bar{E}$, thus $a\bar{E} = \bar{E}a$.

Therefore $\bar{E}$ is a invariant subgroup of N(E).

INFERENCE 1. Let E be a SubSemigroup of G and $e \in E$. If $G_0 < N(E)$, then $G_0 \cap \bar{E} \triangleleft G_0$.

INFERENCE 2. Let E be a Subsemigroup of G and $e \in E$, If $G_0 < N(E)$, then $G_0|_{\bar{E}} = \{a\bar{E} \mid a \in G_0\}$ is a normal hypergroup of $\bar{E}$. In special, if $G_0 \supset \bar{E}$, then $G_0|_{\bar{E}}$ is a quotient group $G_0/\bar{E}$.

LEMMA 2. Let E be a SubSemigroup of G and $e \in E$. If $G_0 < N(E)$, then for any $a, b \in G_0$ we have $aE = bE \Leftrightarrow a\bar{E} = b\bar{E} \Leftrightarrow a(G_0 \cap \bar{E}) = b(G_0 \cap \bar{E})$.

PROOF. Firstly we prove

$aE = bE \Longrightarrow a\bar{E} = b\bar{E} \Rightarrow a(G_0 \cap \bar{E}) = b(G_0 \cap \bar{E})$

As $aE = bE$, from $e \in E$ there is $a^{-1}b \in E$ and $b^{-1}a \in E \Rightarrow a^{-1}b \in \bar{E} \Rightarrow a\bar{E} = b\bar{E}$.

Assume $a(G_0 \cap \bar{\bar{E}}) \neq b(G_0 \cap \bar{\bar{E}})$ there is $a^{-1}b \notin G_0 \cap \bar{\bar{E}}$. From $G_0 < N(E)$ there is $a^{-1}b \in G_0 \Longrightarrow a^{-1}b \notin \bar{\bar{E}} \Longrightarrow a\bar{\bar{E}} \neq b\bar{\bar{E}}$. This is at Variance with $a\bar{\bar{E}} = b\bar{\bar{E}}$, so $a(G_0 \cap \bar{\bar{E}}) = b(G_0 \cap \bar{\bar{E}})$.

Now we prove

$$a(G_0 \cap \bar{\bar{E}}) = b(G_0 \cap \bar{\bar{E}}) \Longrightarrow a\bar{\bar{E}} = b\bar{\bar{E}} \Longrightarrow aE = bE$$

As $a(G_0 \cap \bar{\bar{E}}) = b(G_0 \cap \bar{\bar{E}})$ there is $a^{-1}b \in G_0 \cap \bar{\bar{E}} \subset \bar{\bar{E}} \Longrightarrow a\bar{\bar{E}} = b\bar{\bar{E}} \Longrightarrow a^{-1}b \in \bar{\bar{E}} \subset E$ and $b^{-1}a \in \bar{\bar{E}} \subset E \Longrightarrow b \in aE$ and $a \in bE \Longrightarrow bE \subset aE^2 = aE$ and $aE < bE^2 = bE \Longrightarrow aE = bE$.

Consider two mappings

$f: G_0|_E \longrightarrow G_0|_{\bar{\bar{E}}}$, $f(aE) = a\bar{\bar{E}}$ (for $a \in G_0$).

$g: G_0|_{\bar{\bar{E}}} \longrightarrow G_0/G_0 \cap \bar{\bar{E}}$, $g(a\bar{\bar{E}}) = a(G_0 \cap \bar{\bar{E}})$ (for $a \in G_0$).

For any a, $b \in G_0$ there is

$f((aE)\ (bE)) = f(abE) = ab\bar{\bar{E}} = (a\bar{\bar{E}})(b\bar{\bar{E}})$

$g((a\bar{\bar{E}})(b\bar{\bar{E}})) = g(ab\bar{\bar{E}}) = ab(G_0 \cap \bar{\bar{E}}) = a(G_0 \cap \bar{\bar{E}})\ b(G_0 \cap \bar{\bar{E}})$

Therefore mappings f and g are two homomorphic mappings.

From above results we have

THEOREM 1. Let E be a SubSemigroup of G and $e \in E$. If $G_0 < N(E)$, then $G_0|_E \cong G_0|_{\bar{\bar{E}}} \cong G_0/G_0 \cap \bar{\bar{E}}$. (here the $G_0/G_0 \cap \bar{\bar{E}}$ is a quotient group)

In particular,

i) If $G_0 \cap \bar{\bar{E}} = e$, then $G_0|_E \cong G_0|_{\bar{\bar{E}}} \cong G_0$

ii) If $G_0 \supset \bar{\bar{E}}$, then $G_0|_E \cong G_0|_{\bar{\bar{E}}} = G_0/\bar{\bar{E}}$

iii) If $G_0 = N(E)$, then $N(E)|_E \cong N(E)|_{\bar{\bar{E}}} = N(E)/\bar{\bar{E}}$.

(here the $G_0/\bar{\bar{E}}$ and $N(E)/\bar{\bar{E}}$ are two quotient groups).

EXAMPLE. Let nonzero rational number Q be a group with the multiplication, nonzero whole number Z be a SubSemigroup of Q .

We write

$Q|_Z = \{ aZ \mid a \in Q \}$

So $Q|_Z$ is a normal hypergroup and his nuit element is the Z.

For $Q \cap \bar{Z} = \bar{Z} = \{ -1, 1 \}$
So $Q|_Z \cong Q/\bar{Z}$

here $Q/\bar{Z}$ is a group formed by nonzero positive retional number.

In the algebra we know :

If N is a invariant subgroup of group G, so any Subgroups of the quotient group G/N are K/N (here $K < G$ and $K \supset N$), and the K/N is a invariant Subgtoup of G/N iff k is a invariant Subgroup of G (here $K \supset N$).

From above results we have

THEOREM 2. Let E be a SubSemigroup of G and $e \in E$. If G_0 be Subgroup of N(E), then i) Any Subgroup of the normal hypergroup $G_0|_E$ is:

$G_1|_E = \{ aE \mid a \in G_1, \ G_1 < G_0 \text{ and } G_1 \supset G_0 \cap \bar{E} \}$

ii) Any invariant Subgroup of the normal hypergroup $G_0|_E$ is

$G_1|_E = \{ aE \mid a \in G_1 \ , \ G_1 \lhd G_0 \text{ and } G_1 \supset G_0 \cap \bar{E} \}$

iii). Any invariant Subgroup Series of the normal hypergroup $G_0|_E$ is

$$G_0|_E \supset G_1|_E \supset \cdots\cdots \supset G_{n-1}|_E \supset G_n|_E$$

here $G_0 \supset G_1 \supset - - - - - - \supset G_{n-1} \supset G_n = G_0 \wedge \bar{E}$ is an invariant Subgroup Series of the G_0 .

In the above theorem 2 if $G_0 = N(E)$, then they are Subgroups, invariant Subgroups and invariant Subgroup Series of the greatest normal hypergroup $N(E)|_E$ of the E respectively.

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