Neighborbood structures, Multiple choice principle and relative algebraic problems

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In this paper, we shall expound that the research of topology in fuzzy set framework is helpful to deepen our knowledge of some fundamendal concepts of topology. Moreover, this research can lead to several algebraic problems which seem to be interesting from point of view of algebra.

1. Multiple choice principle (MCP) and neighborhood structures

In the ordinary set theory, the following fact is very fundamental: If a point x belongs to the union of some sets A_j $(j\in J)$, then x also belongs to certain A_j. We call this trivial statement as Multiple choice principle (MCP for short), because it is used almost everywhere in mathematical arguments and as shown as in the sequel it really possesses profound connotation.

What is the vesion of MCP in fuzzy set theory? At first we need to define the notion of "points" (i.e. fuzzy points). Suppose X is non-empty set and L is complete lattice which has the largest element 1 and the smallest element 0. A lattice-valued mapping $A: X \longrightarrow L$ is called L-fuzzy set (F-set for short); If A takes value 0 for all $y\in X$ except one, say $x\in X$, we called A as fuzzy point and denote it by x_b where $b=A(x)$. The collection L^X of all the F-sets is also a complete lattice. For $A_i\in L^X$, the join $\bigvee A_i$ is defined by $(\bigvee A_i)(x) \overset{d}{=} \bigvee A_i(x)$ $(x\in X)$. Similarly we define meet operation. Thus in L^X, $A\leq B \Longleftrightarrow \forall x\in X,\ A(x)\leq B(x)$. Here we note: If L possesses complement operation "'" (namely an order reversing involution), then in L^X, we have also complement, defined as $A'(x)=(A(x))'$ $(\forall x\in X)$.

Now the MCP can be expressed as follows:

if fuzzy point $x_b \leq \bigvee\{A_j\}$, then there is a A_j such that $x_b\leq A_j$.

Consider a simple case: $X=\{x\}$ and $L=[0,1]$. We put $A_j=x_{1-\frac{1}{j+1}}$ $(j=1,2,\ldots)$. Clearly $\bigvee\{A_j\}=x_1$. Hence $x_1\leq \bigvee\{A_j\}$, but $x_1\nleq$ any A_j. That is to say, the MCP is violated in fuzzy set theory (Presicely, it is violated with respect to the above "belonging relation $\leq$"). Now we discuss further the influence on topology affected by the failure of MCP.

In general topology, the topology can be locally defined via open neighborhood systems at points. Namely, the family of open sets to which the point belongs. Namely

Neighborhood structure = some topological condition (e.g. open sets) + A membership relation between a point and a set (e.g. $\in$)

This idea can also apply to fuzzy set theory. Suppose that $(L^X, \mathfrak{T})$ is a fuzzy topological space (where $\mathfrak{T} \subseteq L^X$ is closed under arbitrary join and finite meet and each member of $\mathfrak{T}$ is called fuzzy open set). Then with respect to the above belonging relation "$\leq$", the open neighborhood system of fuzzy point x_b is $\{U \in \mathfrak{T} : x_b \leq U\}$. This neighborhood structure seems to be "natural" and is early (1974) transmitted to fuzzy topology. However, because of the failure of MCP, the coresponding fuzzy topology research leads to many pathological results. In face of the seirous situation, we need to find a reasonable neighborhood structure. In other words, we must give a deeper analysis on fuzzy membership relation between fuzzy point x_b and fuzzy set A. Denote the desired membership relation as $\lhd$. The crucial point which we shall insist is: MCP must be valid with respect to $\lhd$. Namely, $x_b \lhd \bigvee\{A_j\} \Rightarrow \exists\, j \in J,\ x_b \lhd A_j$.

2. Principles to determine the fuzzy membership relation $\lhd$

Now assume the value-set (i.e. range) L possesses complement "'" .

(1) Extension principle. Restricting the relation $\lhd$ to the ordinary set theory, $\lhd$ will become the usuall belonging relation $\in$.

(2) Value-set determination principle The fact whether $x_b \lhd A$ or not is completely determined by a system of formulae about b and $A(x)$ expressed in terms of the order relation $\leq$ and complement "'". Moreover, the system of formulae is valid not only for some b and $A(x)$, but for any complete lattice L with a complement and any $b \in L$, $b \neq 0$ and any $A(x)$ as well.

(3) Maximum and minimum principle For any x_b, $x_b \lhd X$ and x_b not $\lhd\ \emptyset$, where X and $\emptyset$ denote the largest and smallest F-set of L^X respectively.

(4) Multiple choice principle (MCP) Assume $\{A_j\}$ is a family of F-sets, then

$$x_b \lhd \bigvee\{A_j\} \Longrightarrow \text{theres is a } A_j \text{ such that } x_b \lhd A_j .$$

These principles appear to be intuitive and natural, and we can proved that

(i) principles (2) + (3) $\Rightarrow$ principle (1). Namely, the principle (1) may be deleted!

(ii) principles (2), (3) and (4) are independent each other.

3. Q-coincidence relation and Q-neighborhoods

Definition A fuzzy point x_b is said to be Q-coincident with F-set A iff $b \nleq A'(x)$;

Theorem 1 The fuzzy membership relation satisfying the four above-mentioned principles is exaclly the Q-coincident relation.

The proof is referred to : Liu, Y.M., An analysis of fuzzy membership relation in fuzzy set theory, Proc. of Fuzzy Inform., Knowledge Representation and Decision Analysis, 1983, 115-122.

Thus in fuzzy topological space $(L^X, \mathfrak{T})$ a Q-neighborhood system of x_b is introduced as follows: B is open Q-neighborhood of x_b iff $B \in \mathfrak{T}$ and $b \nleq B'(x)$.

After introducing Q-neighborhood system, we have sucessfully establish following:

(1) A satisfactory fuzzy Moore-Smith convergence theory (Pu, B & Liu, Y.M., Fuzzy topology I,II, J. Math. Anal. Appl. 76(1980) 571-599, 77(1980), 20-37.)

(2) Fuzzy compactness theory (Liu, Y.M., Compactness and Tychonoff theorem in fuzzy topological spaces, Acta Math. Sinica, 24(1981), 260-269 (in Chinese) and Wang, G., A new fuzzy compactness defined by fuzzy nets, J. Math. Anal. Appl., 94(1983), 1-23).

(3) Imbedding theory and metrization (Liu, Y.M. A pointwise characterization of completely regularity and imbedding theorem in fuzzy top. spaces, Scientia Sinica 26(1983), 138-147; and Liu, Y.M. Fuzzy metrization—an application of imbedding theory, Chapter 14 in "The Analysis of Fuzzy Information" , Ed. by J. Bezdek, CRC, 1986).

(4) Compactifications theory (Liu, Y.M. & Luo Mao-Kang, Induced spaces and fuzzy Stone-Čech compactifications (to appear); The preorder of fuzzy compactifications (to appear), etc).

(5) Fuzzy function spaces (Peng, Yuwei, Top. structure of fuzzy function spaces --- pointwise convergence topology and compact open topology, Acta Math. Sinica, 28(1985), 799-808. (in Chinese)).

About the significant role of Q-neighborhood system, we also refer to paper: Warner, M.W., Some thoughts on lattice valued functions and relations, in "Aspects of

Topology" Eds. I.M. James et all, Cambridge Univ. Press, 1985), 127-140.

Now we turn to investigate the general version of MCP and its varieties. Some relative algebraic problems are also discussed.

4. MCP for lattices

Notice that in the above discussion both the fuzzy point and F-set A are elements of L^X. If we consider the general version of MCP for lattice, then only elements of lattice are involved. Thus we have the following version of MCP for complete lattice L:

For any $b \in L$ and $b_j \in L$ $(j \in J)$. $b \leq \bigvee_{j \in J} \{b_j\} \Rightarrow b \leq$ certain b_j ... (*)

5. Finite MCP

When J is finite index set, formula (*) become finite MCP, meanwhile we say b is a molecule or irreducible element. L is said to be a molecular lattice, if for any $b \in L$, b is join of some molecules.

Theorem 2 A complete lattice L is a molecular lattice iff it is isomorphic (with respect to arbitrary meet and, finite join) to the lattice of all closed sets of a topological space.

In non-atomic lattices, the irreducible elements (or molecules) are a suitable substitutes of points (or atoms), hence they are important. About top. molecular lattices, see Wang, Guojun, Generalized topological molecular lattices, Scientia Sinica, 28(1984), 785-798.

6. A variety of MCP — directed pairwise MCP

In continuous lattices theory (see Gierz, G. et al, A compendium of continuous lattices, 1980), the central concept is Way-below relation, denoted by $\ll$. Let L be a lattice, $a,b \in L$, we say $a \ll b$, if for any directed set $D \subseteq L$, $b \leq \bigvee D \Rightarrow \exists\, d \in D$ with $a \leq d$. A complete lattice is said to be a continuous lattice (an algebraic lattice), if $\forall a \in L$, $a = \bigvee \{b \in L: b \ll a\}$ $(a = \bigvee \{b \in L: b \ll b \leq a\})$

The abundant results on continuous lattices and algebraic lattices may be referred the above book by Gierz. Here we emphasize that the way-below relation $a \ll b$ is a kind of MCP for pair (a,b). (or pairwise MCP). Meantime the way-below relation is also a kind of directed MCP, namely the relative subfamily $D \subseteq L$ is required to be directed set. In a word, the way-below relation is a directed pairwise MCP.

7. Pairwise MCP

We investigate pairwise MCP further. Assume L be a complete lattice, $b, c \in L$. We say c is a point of b, in symbol $c \leftarrow b$, if for any $E \subseteq L$, $b \leq \bigvee E \Rightarrow \exists e \in E$ such that $c \leq e$.

As known as well, a subset $M \subseteq L$ is called a minimal set of b, iff (1) $b = \bigvee M$ and (2) for any $E \subseteq L$ with $b \leq \bigvee E$ and for any $m \in M$, there is $e \in E$ such that $m \leq e$. (Liu, Y.M., Intersection operation on Union-preserving mapping in completely distributive lattices, J. Math. Anal. Appl., 84(1981), 249-255).

Clearly $M \subseteq L$ is a minimal set of b iff (1) $b = \bigvee M$, and (2) $\forall\, m \in M$ is a point of b

In a sense, a minimal set of b is a pointlike descomposition of b. Therefore this concept is rather useful in lattices theory, see Liu, Y.M. & He, Ming, Induced mappings in completely distributive lattices, Proc. 15th Inter. Symp. of Multiple-valued logic, IEEE, 1985, 346-353. Here we shall point out the following

Theorem 3 Let L be a complete lattice. L is completely distributive lattice iff each element of L have its minimal set.

He, Ming have recently defined the pointed lattice L as a complete lattice and for $\forall\, b \in L$, $b = \bigvee \{c \in L,\ c \leftarrow c \leq b\}$. He also called the $c \in L$ a point of L iff $c \leftarrow c$. Moreover, He also give several interesting results of the pointed lattices, such as the following:

Theorem 4 A lattice is a pointed lattice iff it is an algebraic + molecular.

Theorem 5 L is a pointed lattice iff it is isomorphic to a sub-complete lattice of some power set $\mathcal{P}(X)$ (Note: Here isomorphism is defined with respect to arbitrary join and arbitrary meet).