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NECESSITY MEASURES AND THE RESOLUTION PRINCIPLE

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Abstract

In this paper a careful distinction is made between fuzzy propositions (i.e. propositions involving vague predicates) which may have intermediary degrees of truth, and uncertain propositions (with non-vague predicates) whose truth or falsity cannot definitely established due to the incompleteness of the available information. Then the resolution principle is extended in the case of uncertain propositions where the uncertainty is modelled in terms of necessity measures. The alternative use of probability measures or of Shafer's belief functions is also discussed.

Key words : uncertainty ; degree of truth ; necessity measure ; possibility measure ; probability ; fuzzy set ; resolution principle ; inference.

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1 - Introduction and background

1) Uncertain propositions

The management of uncertainty in inference systems is an important issue, due to the imperfect nature of real world information. Recently a new approach, based on possibility and necessity measures, has emerged for the treatment of uncertain pieces of knowledge. In the following, the basic background of this approach is given.

Let $(P,\vee,\wedge,\neg)$ be a Boolean algebra of propositions. A possibility measure Π defined on P satisfies the following axioms [18]

$$\Pi(\mathbb{0}) = 0 \; ; \; \Pi(\mathbb{1}) = 1$$
$$\forall p, \forall q, \Pi(p \vee q) = \max(\Pi(p),\Pi(q)) \quad , \qquad (1)$$

where $\mathbb{0}$ (resp. $\mathbb{1}$) denotes the ever-false (resp. ever-true) proposition ; i.e. $\forall p \in P$, $p \wedge \neg p = \mathbb{0}$ and $p \vee \neg p = \mathbb{1}$. $\Pi(p)$, which belongs to the real interval [0,1] is an estimate of the degree of possibility that the proposition p is true. Note that $\Pi(p)$ is not a degree of truth ; here we only have the two usual truth-values : 'true' and 'false'. Moreover, as a direct consequence of (1) we have

$$\forall p, \max(\Pi(p),\Pi(\neg p)) = 1 \quad , \qquad (2)$$

which expresses that at least one of two opposite propositions must be considered as possibly true. By duality a necessity measure N is associated with a possibility measure Π according to a definition [2] which extends the usual relationship between possibility and necessity in modal logic,

$$\forall p, N(p) = 1-\Pi(\neg p) \qquad (3)$$

Clearly N satisfies the axioms

$$N(\mathbb{0}) = 0 \; ; \; N(\mathbb{1}) = 1$$
$$\forall p, \forall q, N(p \wedge q) = \min(N(p),N(q)) \; . \qquad (4)$$

We have $\forall p, \min(N(p),N(\neg p)) = 0$. (5)

$N(p)$ is the extent to which the proposition p can be considered as necessarily (or certainly) true with respect to the current state of knowledge ; note that, due to (5), as soon as $N(p) > 0$, then $N(\neg p) = 0$, i.e. two opposite propositions cannot be simultaneously considered as somewhat certainly true. When the proposition p is known or proved to be true, we have $N(p) = 1$ (or equivalently $\Pi(\neg p) = 0$) ; it entails $\Pi(p) = 1$, but $\Pi(p) = 1$ is not a sufficient condition for asserting that p is true. When the proposition p is known or proved to be false, $\neg p$ is true and we have $N(\neg p) = 1$ (or equivalently $\Pi(p) = 0$).

Note that we only have the following inequalities

$$\forall p, \forall q, N(p \vee q) \geq \max(N(p), N(q)) \quad (6)$$

$$\forall p, \forall q, \Pi(p \wedge q) \leq \min(\Pi(p), \Pi(q)) \quad (7)$$

The corresponding equalities do not hold in general. Besides, it can be proved that if $p \to q = \mathbb{T}$ then $\Pi(q) \geq \Pi(p)$ and $N(q) \geq N(p)$, where $p \to q$ stands for

Then the following patterns of reasoning can be easily validated (see [14], [4], [6])

$$\frac{\begin{array}{ll} N(p \to q) & = \alpha \\ N(p) & = \beta \end{array}}{\alpha \geq N(q) \geq \min(\alpha, \beta)} \quad (8)$$

and

$$\frac{\begin{array}{ll} N(p \to q) & = \alpha \\ N(\neg q) & = \beta \end{array}}{\alpha \geq N(\neg p) \geq \min(\alpha, \beta)} \quad (8')$$

where $p \to q$ is defined as $\neg p \vee q$ (which entails $p \to q = \neg q \to \neg p$). The patterns (8) and (8') respectively extend the modus ponens and the modus tollens, which are recovered for $\alpha = \beta = 1$. Note that there does not exist an analogous pattern of reasoning changing the necessity measure into a possibility measure everywhere and min into another aggregation operation ; see [14], [4], [6]. Moreover we can extend necessity measures to formulas involving predicates, by postulating that

$$N(\forall x\, P(x)) = \inf_{x \in D} N(P(x)) \quad (9)$$

where P is a predicate and D is the domain of variable x. Note that (9) is in agreement with (4) when D is finite, since then $\forall x P(x)$ is equivalent to the conjunction $P(a_1) \wedge \ldots \wedge P(a_n)$ where $D = \{a_1, \ldots, a_n\}$. Then, using (3), we have

$$\Pi(\exists x P(x)) = \sup_{x \in D} \Pi(P(x)) \tag{10}$$

which in turn is in agreement with (1). Thus the following pattern of reasoning is valid

$$\frac{N(\forall x P(x)) = \alpha}{N(P(a)) \geq \alpha} \tag{11}$$

which extends the usual particularization mode of inference.

Possibility and necessity measures are an alternative to probability measures for representing uncertainty ; they enable us to distinguish between the total lack of certainty in the truth of p ($N(p) = 0$) and the total certainty that p is false ($\Pi(p) = 0$).

2) Fuzzy propositions

By "fuzzy proposition" we mean a proposition which involves a vague predicate or a vague quantifier. In the following we only consider propositions with vague predicates. For instance 'John is tall' is a fuzzy proposition since the meaning of 'tall' remains vague, even in a given context. A fuzzy proposition may have a degree of truth which is intermediary between 'true' and 'false'. For instance, let h(John) be the value of John's height, supposed to be precisely known, then $\mu_{tall}(h(John))$ is a number belonging to the real interval [0,1] which can be viewed as the degree of truth of the proposition 'John is tall' (μ_{tall} is the membership function representing the fuzzy set of heights regarded as tall in a given context) ; see [17]. The truth-value v of a compound fuzzy proposition p can be expressed in terms of the truth-values of its components, according to the following formulas [17]

$$v(\neg p) = 1 - v(p) \tag{12}$$

$$v(p \wedge q) = \min(v(p), v(q)) \tag{13}$$

$$v(p \vee q) = \max(v(p), v(q)) \tag{14}$$

This can be extended to propositions with universal and existential quantifiers, using the formulas

$$v(\forall x\ P(x)) = \inf_{x \in D} v(P(x)) \tag{15}$$

$$v(\exists x\ P(x)) = \sup_{x \in D} v(P(x)) \tag{16}$$

The use of conjunctive and disjunctive operators other than min and max, as well as other complementation operators can be contemplated ; see [2] for instance. With definitions (12)-(14), the excluded middle law and the contradiction law no longer hold for fuzzy propositions.

Uncertain propositions, considered in the preceding subsection, must not be confused with fuzzy propositions. In the first case we have propositions which are true or false (thus involving non-vague predicates), but due to the lack of precision of the available information we can in general only estimate to what extent it is possible or necessary that a proposition is true. In the second case the available information is precise, but the vagueness of predicates leads to have propositions with intermediary degrees of truth. Obviously we may encounter a fuzzy proposition for which the available reference information is not precise ; then we have the general case of an uncertain fuzzy proposition ; the study of such propositions is out of the scope of the present note ; see [3] for a discussion. Thus the apparent similarity between (3), (4), (1), (9) and (10) on the one hand and (12), (13), (14), (15) and (16) on the other hand, due to the common use of the operators 1-(.), min and max, is superficial and must not lead us astray.

Patterns of reasoning in the style of (8) can be developed for fuzzy propositions, i.e. compute bounds for v(q) from the knowledge of v(p) and $v(p \to q)$; see [2] for instance. However there are different natural ways for defining $v(p \to q)$ from (12)-(14), as discussed in [5] ; moreover we may have $v(\neg p \to \neg q) \neq v(p \to q)$, for some definitions of the implication operator, when $v(p \to q)$ is not defined as $v(\neg p \vee q)$. It can be proved that we have $v(q) \geq \min(v(\neg p \vee q), v(p))$ only if $v(\neg p \vee q) \geq 0.5$ and $v(p) > 0.5$ (see [2], p.167).

Quite early in the development of fuzzy set theory, an extension of Robinson's resolution principle [15] was proposed by Lee [9] for ground clauses in the framework of the fuzzy logic defined by (12)-(14) i.e. for dealing with fuzzy propositions ; note that the resolution principle avoids the explicit use of the implication connective in the representation of the knowledge. Basically, Lee [9] proved that if all the truth-values of parent clauses are greater than 0.5, then a resolvent clause derived by the resolution principle always has a truth-value between the maximum and the minimum of those of the parent clauses. This result was discussed and completed by Aronson, Jacobs and Minker [1] and by Mukaidono [11]; Lee's result has been recently used by Ishizuka and Kanai [8] and by Orci [12] for developing fuzzy logic programming languages. See also Martin [10] for the treatment of fuzzy truth values.

In the following, we discuss the extension of the resolution principle to uncertain propositions in the sense of the preceding subsection.

II - Resolution principle for uncertain propositions

1) Ground clauses

It is assumed that the reader is familiar with the resolution principle. In its simplest form, the resolution principle corresponds to the following pattern of reasoning in the propositional case

$$\frac{\vdash p \vee q \qquad \vdash \neg p \vee r}{\vdash q \vee r} \qquad (17)$$

$p \vee q$ and $\neg p \vee r$ are called the parent clauses and $q \vee r$ is the resolvent of this pair of clauses. The pattern (17) can be generalized to uncertain propositions under the form

$$\frac{N(p \vee q) = \alpha \qquad N(\neg p \vee r) = \beta}{N(q \vee r) \geq \min(\alpha, \beta)} \qquad (18)$$

For $\alpha = \beta = 1$, the pattern (17) is recovered. Note that the values of $N(p \vee q)$ and of $N(\neg p \vee r)$ can be independently assigned except if $q = r = 0$, which contrasts with the case of fuzzy propositions, where the assignment $v(p \vee q) = \alpha$, $v(\neg p \vee r) = \beta$ requires $\alpha \geq 1-\beta$. The pattern (18) is easy to establish since using (4), we have $N(q \vee r) = N(p \vee q \vee r) \wedge (\neg p \vee q \vee r)) = \min(N(p \vee q \vee r), N(\neg p \vee q \vee r))$. Then using (6), we have $N(p \vee q \vee r) \geq N(p \vee q) = \alpha$ and $N(\neg p \vee q \vee r) \geq N(\neg p \vee r) = \beta$, which makes the result obvious. Besides, no non-trivial upper bound can be found for $N(q \vee r)$ in (18), which differs from the result obtained by Lee [9] in terms of truth-values of fuzzy propositions. Indeed, it can be checked using (4) that the assignment $N(p \vee q \vee r) = N(\neg p \vee q \vee r) = 1$, $N(p \vee q \vee \neg r) = \alpha$ and $N(\neg p \vee \neg q \vee r) = \beta$ gives $N(q \vee r) = 1$ while $N(p \vee q) = \alpha$ and $N(\neg p \vee r) = \beta$; the above assignment is feasible, since due to the axiom (4) a necessity measure on the Boolean algebra generated by $\{p,q,r\}$ is defined by its value on the clauses with three distinct literals.

It is well-known that in classical logic the resolution principle is an inference rule which encompasses the modus ponens, the modus tollens and the chaining. This is still true in case of uncertainty. Indeed with $q = \emptyset$, the pattern (18) reduces to (8), and with $r = \emptyset$, it reduces to (8') since $p \vee q = \neg q \rightarrow p$; changing q into $\neg q$ in (18) yields $N(q \rightarrow r) \geq \min(\alpha,\beta)$ when $N(q \rightarrow p) = \alpha$ and $N(p \rightarrow r) = \beta$. However the upper bound on the certainty of the conclusion, which can derived in the extended modus ponens (8) or modus tollens (8'), can no longer be obtained with (18). Note that this upper bound is brittle, since with $N(p \rightarrow q) \geq \alpha$ and $N(p) \geq \beta$ we only conclude $N(q) \geq \min(\alpha,\beta)$; i.e. no non-trivial upper bound exists for $N(q)$; a similar result holds with (8') . Obviously, from $N(p \vee q) \geq \alpha$ and $N(\neg p \vee r) \geq \beta$, (18) enables us to deduce $N(q \vee r) \geq \min(\alpha,\beta)$.

Let S be a set of ground clauses. By R(S) we mean the union of S with the set of all ground clauses obtainable from S using one application of the resolution principle (i.e. all the resolvents of the pairs of members of S). Let $R^n(S)$ be result of iterating the rule n times. Then, due to (18) and the associativity of the min operation, we can state the following

<u>Theorem</u> : Let $S = \{C_1,\ldots,C_m\}$. Let $\forall\ i = 1,m$, $N(C_i) = \alpha_i$. Let C^n denote any clause in $R^n(S)$. Then $\forall\ n \geq 0$, $N(C^n) \geq \min_{i=1,m} \alpha_i$. This theorem expresses that the degree of certainty (expressed in terms of necessity) of any logical con-

trasts with the case of fuzzy propositions, where the assignment $v(p \vee q) = \alpha$, $v(\neg p \vee r) = \beta$ requires $\alpha \geq 1-\beta$. The pattern (18) is easy to establish since using (4), we have $N(q \vee r) = N((p \vee q \vee r) \wedge (\neg p \vee q \vee r)) = \min(N(p \vee q \vee r), N(\neg p \vee q \vee r))$. Then using (6), we have $N(p \vee q \vee r) \geq N(p \vee q) = \alpha$ and $N(\neg p \vee q \vee r) \geq N(\neg p \vee r) = \beta$, which makes the result obvious. Besides, no non-trivial upper bound can be found for $N(q \vee r)$ in (18), which differs from the result obtained by Lee [9] in terms of truth-values of fuzzy propositions. Indeed, it can be checked using (4) that the assignment $N(p \vee q \vee r) = N(\neg p \vee q \vee r) = 1$, $N(p \vee q \vee \neg r) = \alpha$ and $N(\neg p \vee \neg q \vee r) = \beta$ gives $N(q \vee r) = 1$ while $N(p \vee q) = \alpha$ and $N(\neg p \vee r) = \beta$; the above assignment is feasible, since due to the axiom (4) a necessity measure on the Boolean algebra generated by $\{p, q, r\}$ is defined by its value on the clauses with three distinct literals.

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sequence obtained by repeatedly applying the resolution principle, will be at least equal to the one of the most uncertain parent clause. This simple result agrees with our intuition. Note that some C^m such that $N(C^n) > \min_{i=1,m} \alpha_i$ exist.

Remark 1 : A similar result exists when the uncertainty is modelled in terms of probability. Indeed the following pattern can be easily established

$$\frac{\begin{array}{l}\text{Prob}(p \vee q) = \alpha \\ \text{Prob}(\neg p \vee r) = \beta\end{array}}{\text{Prob}(q \vee r) \geq \max(0, \alpha+\beta-1)} \qquad (19).$$

First, note that we must have $\alpha+\beta \geq 1$ in (19) since we have necessarily $\text{Prob}(p) \leq \alpha$ and $\text{Prob}(\neg p) \leq \beta$. Besides no non-trivial upper bound exists, since it can be checked that $\text{Prob}(q \vee r) = 1$ is compatible with the premises of (19). To prove (19), let us add the members of the two equalities

$$\alpha = \text{Prob}(p \vee q) = \text{Prob}(p) + \text{Prob}(q) - \text{Prob}(p \wedge q)$$

$$\beta = \text{Prob}(\neg p \vee r) = \text{Prob}(\neg p) + \text{Prob}(r) - \text{Prob}(\neg p \wedge r)$$

it yields

$$\alpha+\beta- 1 = \text{Prob}(q \vee r) + \text{Prob}(q \wedge r) - \text{Prob}((p \wedge q) \vee (\neg p \wedge r))$$

whence the result since $(p \wedge q) \vee (\neg p \wedge r) = (p \wedge q) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) = (q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$ and if $p \rightarrow q = \mathbb{1}$ then $\text{Prob}(q) \geq \text{Prob}(p)$.

The operation $*$ defined by $\alpha * \beta = \max(0, \alpha+\beta-1)$ on $[0,1] \times [0,1]$ is associative and is such that $\alpha *^m \alpha \stackrel{\Delta}{=} \alpha * \alpha * \ldots * \alpha$ (m times α) $= \max(0, m.\alpha-m+1)$ and thus $\forall \alpha \neq 1$, $\exists M$ finite, $\forall m \geq M$, $\alpha *^m \alpha = 0$. As a consequence, the repeated application of (19) may lead to a lower bound which is not very much informative, whatever the quality of the lower bounds on the probabilistic degrees of certainty of the parent clauses. This behavior does not exist with min. Moreover we have $\forall (\alpha,\beta) \in [0,1]^2$, $\min(\alpha,\beta) \geq \max(0,\alpha+\beta-1)$ which compensates the fact that $N(p)$ must be regarded in general as a lower bound of $\text{Prob}(p)$; see [2] for instance, p. 138.

Remark 2 : It is worth noticing that the pattern (19) still holds for Shafer's belief functions [16], i.e.

$$\frac{\begin{array}{l}Bel(p \vee q) = \alpha \\ Bel(\neg p \vee r) = \beta\end{array}}{Bel(q \vee r) \geq \max(0, \alpha+\beta-1)} \tag{20}$$

where Bel is a function from $\mathcal{P}$ to [0,1], which can be built from a so-called basic probability assignment m (a function from $\mathcal{P}$ to [0,1] such that $m(\mathbb{0}) = 0$ and $\sum_{p \in \mathcal{P}} m(p) = 1$) according to the formula [16]

$$Bel(p) = \sum_{q : q \to p = \mathbb{1}} m(q) \tag{21}$$

Using (21), (20) is easy to prove. Indeed, we have

$$\alpha+\beta = Bel(p \vee q) + Bel(\neg p \vee r) = \sum_{s \in C_1} m(s) + \sum_{s \in C_2} m(s)$$

with $C_1 = \{s \in \mathcal{P}, \neg s \vee p \vee q = \mathbb{1}\}$ and $C_2 = \{s \in \mathcal{P}, \neg s \vee \neg p \vee r = \mathbb{1}\}$

Let $C_3 = \{s \in \mathcal{P}, \neg s \vee [(p \vee q) \wedge (\neg p \vee r)] = \mathbb{1}\} = C_1 \cap C_2$. Then we get

$$\alpha+\beta = \sum_{s \in C_1 \cup C_2} m(s) + \sum_{s \in C_3} m(s)$$

$$\leq \sum_{s \in \mathcal{P}} m(s) + \sum_{\neg s \vee (q \vee r) = \mathbb{1}} m(s) = 1 + Bel(q \vee r)$$

since $C_3 \subseteq \{s \in \mathcal{P}, \neg s \vee (q \vee r) = \mathbb{1}\}$. Necessity measures and probability measures are particular cases of belief functions [2] ; the pattern (20) shows that the lower bound obtained with probability measures is the same as with any belief functions ; this lower bound is improved in case of necessity measures.

2) Predicates

The resolution principle for predicate calculus can be stated in the following way. Let L_1 be an atomic formula, i.e. a predicate symbol of degree n followed by n terms (a constant is a term, a variable is a term, a function bearing on terms is still a term). Let L_2 be the negation of the same predicate with different terms. Let q and r be clauses. Let $p[\sigma]$ denote the clause obtained by applying the set of elementary substitutions specified by σ to the occurrences of variables in the clause p. If the elementary substitution in σ_1, applied to the variables in L_1 and L_2, make L_2 identical to $\neg L_1$, then from $L_1 \vee q$ and $L_2 \vee r$ the resolvent $(q \vee r)[\sigma_1]$ can be deduced.

For example the resolution principle applied to the clauses

$$P(x,f(y)) \vee Q(x) \vee R(a,y)$$
$$\neg P(f(b),z) \vee R(z,t)$$

yields

$$Q(f(b)) \vee R(a,y) \vee R(f(y),t)$$

using the elementary substitutions $f(b)/x$ and $f(y)/z$.

We already observed that the substitution of a variable by a constant in a universally quantified proposition can only increase the necessity degree attached to the proposition (see pattern (11)). More generally from $N(\forall x\ P(x)) \geq \alpha$ we can infer that $\forall y,\ N(\forall y\ P(f(y))) \geq \alpha$ where f is mapping ; note that $N(\forall y\ P(f(y)))$ may be greater than $N(\forall x\ P(x))$ since f is not necessarily onto. Thus the application of the resolution principle for predicate calculus is compatible with a computation of a lower bound of the necessity degree attached to the resolvent using (18) and (11).

For instance if we know that

$$N(\exists x\ P(x)) = \alpha$$

and that $\quad N(\forall y\ P(y) \rightarrow Q(y)) = \beta$

this can be written in a logic programming style using a Skolem constant A, putting the greatest known lower bound of the necessity degree between parentheses after the clause

$P(A) \qquad (\alpha)$

$\neg P(y) \vee Q(y) \qquad (\beta)$

from which we infer (applying the substitution A/y) that

$Q(A) \qquad (\min(\alpha,\beta))$

i.e. $N(\exists z\ Q(z)) \geq \min(\alpha,\beta)$. This very simple example is considered by Nilsson [12] with a probabilistic modelling of uncertainty, but is not dealt with by resolution.

3 - Refutation

A very popular way of using the resolution principle is the refutation method, i.e. the proposition to be proved is assumed to be false, and its negation is added to the set of ground clauses ; when the proposition is actually true, the resolution principle enables the empty clause to be derived, thus establishing a contradiction. The refutation method can provide conclusions which could not be derived by direct application of the resolution principle. Such conclusions q are such that $\exists$ p, $p \to q = \mathbb{I}$ and p can be obtained via the resolution principle.

The refutation method can be extended to the case of uncertain propositions. To do so, the negation of the proposition to prove is added to the set of ground clauses, with a necessity degree equal to 1. For instance if $S = \{\neg p \vee q, p\}$ where $N(\neg p \vee q) = \alpha$, $N(p) = \beta$, and q is to be proved then we start with

$\neg q$	(1)
$\neg p \vee q$	(α)
p	(β)

By resolution we successively obtain

$\neg p$	$(\min(1,\alpha))$
$\emptyset$	$(\min(\alpha,\beta))$

where $\emptyset$ denotes the empty clause, and we conclude that $N(q) \geq \min(\alpha,\beta)$. Indeed the following result holds

Theorem : The grade of necessity attached to the empty clause, corresponds to a lower bound of the grade of necessity of the proposition to prove, using the refutation method for ground clauses .

Proof : Let q be the proposition to prove. Either q can be produced by iterating the resolution principle on the set S of ground clauses, or not. If it can be, we get sooner or later the pattern

$\neg q$ (1)

q (α)

for some α, which leads to

$\square$ (α)

i.e. the lower bound of $N(\square)$ is the same as that of $N(q)$. Otherwise some implicant r of q (i.e. $r \to q = \mathbb{1}$) is obtained, then $\exists$ s, $q = r \vee s$ and the pattern

$\neg r$ (1)

$\neg s$ (1)

r (α)

for some α, leads to

$\square$ (α)

and so, $N(q) \geq N(r) \geq \alpha$. Q.E.D.

Lastly, the method initially proposed by Green[7] for question-answering systems can be adapted with necessity degrees. Namely if the query '$\exists$ x Q(x)?' is to be processed, the clause $\neg Q(x) \vee answer(x)$ is taken for granted instead of $\neg Q(x)$. Instead of deriving the empty clause, the predicate 'answer' is derived, where x may have been substituted. 'answer' serves as a collector capturing a constant a such that Q(a) is hopefully true. For instance, knowing that

$\neg P(x) \vee Q(x)$ (α)

$P(a)$ (β)

in order to answer the query '$\exists$ z Q(z) ?' we assume

$\neg Q(z) \vee answer(z)$ (1)

and by resolution we obtain the result

answer (a) ($\min(\alpha,\beta)$).

III - Concluding remarks

Estimating uncertainty in terms of necessity degrees is compatible with the application of the resolution principle and deriving lower bounds of the necessity degrees of the resolvent is very simple. Contrastedly with the probabilistic case, the obtained lower bounds do not decrease rapidly towards 0 in case of a repetitive use of the resolution principle.

Besides in case of propositions which would be both fuzzy and uncertain, we may think of combining the result obtained by Lee [9] and the results presented here, expressing in terms of a necessity measure to what extent we are sure that the degree of truth of a proposition is greater that some lower bound.

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