A DECISION MODEL UNDER GENERAL INFORMATION

PART II: DECISION RULES

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1. INTRODUCTION.-

In the previous part of this work [2], we studied some aspects of Dempster-Shafer's Theory of Evidence; particularly, the properties of upper and lower integrals were widely considered. In this paper, such results are applied to the problem of decisions making under an information represented as a basic probability assignment.

In our model, we consider a decision problem determined by the following element:

1) Ω The set of nature states.

2) $\mathfrak{D}$ The set of possible decisions.

3) r A recompense function from $\mathfrak{D} \times \Omega$ to $\mathbb{R}$.

4) I An information about Ω given in terms of a basic probability assignment.

We consider Ω and $\mathfrak{D}$ as finite sets in order to avoid convergence and mesurability problems, strange to our objective.

In these conditions, the function r can be expressed by an array in the following way:

$\mathfrak{D} \backslash \Omega$	w_1	w_2	...	w_j	...	w_n
d_1	r_{11}	r_{12}	...	r_{1j}	...	r_{1n}
d_2	r_{21}	r_{22}	...	r_{2j}	...	r_{2n}
	⋮	⋮		⋮		⋮
d_i	r_{i1}	r_{i2}	...	r_{ij}	...	r_{in}
	⋮	⋮		⋮		⋮
d_m	r_{m1}	r_{m2}	...	r_{mj}	...	r_{mn}

We shall denote by $\tilde{r}_i$ the vector ($r_{i1}, r_{i2}, \ldots, r_{ij}, \ldots, r_{in}$).

Then, the problem is to determine the best decision; to solve it, we need to order the vectors $\tilde{r}_i \in \mathbb{R}^n$, using the information I about Ω. An indirect method can be developped: each vector $\tilde{r}_i \in \mathbb{R}^n$, is applied on a real value r_i by means of a function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$f(\tilde{r}_i) = f(r_{i1}, \dots, r_{in}) = r_i \in \mathbb{R}$$

and we say

$$d_i \gtrsim d_k \Longleftrightarrow f(\tilde{r}_i) \geqslant f(\tilde{r}_k) \Longleftrightarrow r_i \geqslant r_k$$

The natural partial order in $\mathbb{R}^n$ must be respected by f, which supposes to verify

$$r_{ij} \geqslant r_{kj} \quad \forall j \Longrightarrow f(\tilde{r}_i) \geqslant f(\tilde{r}_k)$$

We carry out this procedure in two steps. Firstly, we map each vector in $\mathbb{R}^n$ to one in $\mathbb{R}^2$ in the following way:

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}^2$$
$$g(\tilde{r}_i) = \left[I_*(\tilde{r}_i/m) \ , \ I^*(\tilde{r}_i/m) \right]$$

where I_* and I^* are the lower and upper integrals defined by Dempster (1967).

The second step is given by the application of the bidimensional vector to a real number by means of a value-function:

$$v : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$v\left[g(\tilde{r}_i) \right] = r_i \in \mathbb{R}$$

As we shall see below, v reflects the decisor's personal attitude in presence of risk.

2. LOWER AND UPPER RISK.-

We collect here only some properties. A more extensive study can be seen in [2]. Considering the previous definition:

$$g(\tilde{r}_i) = \left[I_*(\tilde{r}_i/m) \ , \ I^*(\tilde{r}_i/m) \right]$$

and taking into account the basic properties of I_* and I^*, we can affirm:

1) $I_*(\tilde{r}_i/m) \leq I^*(\tilde{r}_i/m) \qquad \forall \tilde{r}_i \in \mathbb{R}^n$

2) $I_*(\tilde{r}_i/m)$ assigns to the vector $\tilde{r}_i$ (and, extensively, to d_i) , a value equal to

the mathematical expectation of $\tilde{r}_i$, with respect to the probability measure resulting from the assignation for all the subsets $A \subset \Omega$ of the mass of evidence m(A) - to the element $w_k \in A$ with smaller recompense. So $I_*(\tilde{r}_i/m)$ can be interpreted as the most pessimistic average of the expected recompense from the available information.

3) Similarly, $I^*(\tilde{r}_i/m)$ can be seen as an optimistic evaluation.

4) If $m_1 \subset m_2$ (the evidence m_1 is " included " in m_2; see [2]), then

$$I_*(\tilde{r}_i/m_1) \leq I_*(\tilde{r}_i/m_2)$$
$$\text{and} \quad I^*(\tilde{r}_i/m_1) \geq I^*(\tilde{r}_i/m_2) \qquad \forall \tilde{r}_i \in \mathbb{R}^n$$

Finally, $[I_*(\tilde{r}_i/m) \; , \; I^*(\tilde{r}_i/m)]$ can be considered as a "risk interval" whose length decreases with a better information [4].

3. DECISION RULES BASED ON THE RISK INTERVAL.-

The original problem on $\mathbb{R}^n$ has been transformed into another problem on $\mathbb{R}^2$ by means of the aplication g. So, our task will be to establish an order in $\mathbb{R}^2$. If we consider only the natural and lexicografical orders, the following criterions (rules) are obtained.

i) Dominance criterion.

We will say d_i dominates to d_k according to g, if and only if:

$$I_*(\tilde{r}_i/m) \geq I_*(\tilde{r}_k/m)$$
$$I^*(\tilde{r}_i/m) \geq I^*(\tilde{r}_k/m)$$

denoting by $d_i \succ_g d_k$ this relation.

If one of this inequalities is strictly verified, then we will say d_i dominates strictly to d_k.

ii) Lexicografical order.

Taking into account the decisor's attitude in presence of risk, we consider two orders:

a) Direct. If the decisor doesn't like the risk, then he will try to maximize - the lower bound of his expected recompense; in this way, he will establish the - following relation of preferences:

$$d_i >_L d_k \Longleftrightarrow \begin{cases} I_*(\tilde{r}_i/m) > I_*(\tilde{r}_k/m) & \text{or} \\ I_*(\tilde{r}_i/m) = I_*(\tilde{r}_k/m) \text{ and } & I^*(\tilde{r}_i/m) > I^*(\tilde{r}_k/m) \end{cases}$$

b) Inverse. If the decisor is optimistic, he will pay attention mainly to the -- upper bound of the risk interval (wich is his greater possible recompense). So, he will use the relation of preferences defined by:

$$d_i \gtrdot_{\urcorner} d_k \Longleftrightarrow \begin{cases} I^*(\tilde{r}_i/m) > I^*(\tilde{r}_k/m) & \text{or} \\ I^*(\tilde{r}_i/m) = I^*(\tilde{r}_k/m) \text{ and } & I_*(\tilde{r}_i/m) > I_*(\tilde{r}_k/m) \end{cases}$$

It is inmediate to prove:

$$d_i >_g d_k \Longleftrightarrow \begin{cases} 1) & d_i >_L d_k \\ 2) & d_i \gtrdot_{\urcorner} d_k \end{cases}$$

4. DECISION RULES BASED ON A VALUE FUNCTION.

As it is said at the introduction, we can order $\mathfrak{D}$ by means of a function:

$$v : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Then, we shall define:

$$d_i \gtrsim d_k \Longleftrightarrow v\left[I_*(\tilde{r}_i/m)\ ,\ I^*(\tilde{r}_i/m)\right] \geq v\left[I_*(\tilde{r}_k/m)\ ,\ I^*(\tilde{r}_k/m)\right]$$

In this way, different rules are obtained from different functions v.

Whatever be v, it must verify the following relation:

$$a_1 \geq b_1\ ,\ a_2 \geq b_2 \Longrightarrow v(a_1,a_2) \geq v(b_1,b_2)$$

to maintain a coherent dominance criterion:

$$d_i \gtrsim d_k \Longleftrightarrow f(\tilde{r}_i) \geq f(\tilde{r}_k) \Longleftrightarrow d_i >_v d_k .$$

Among the functions verifying the above relation, can be considered the fo llowing ones:

- $v^*(a,b) = b$; it associates to the interval (I_*,I^*) its upper bound I^*.
- $v_*(a,b) = a$; it assigns to the interval (I_*,I^*) its lower bound I_*.
- $v_c(a,b) = c_1 a + c_2 b$ (with $c_1,c_2 \geq 0$); a combination of I_* and I^* is given from

(I_*, I^*). In the case c_1 and c_2 are normalized, v_c would be a convex combination of v^* and v_*:

$$v_\alpha (I_*, I^*) = \alpha I^* + (1-\alpha) I_*$$

where $\alpha \in [0,1]$ denote the decisor's degree of optimism.

Let us now consider two particular and extreme cases of evidence, in order to show the relation between the application of the above value functions and - the most important classical rules of decision.

a) Probabilistic evidence.

If m_p represents a probabilisty distribution, we have:

$$I_*(\tilde{r}_i/m_p) = I^*(\tilde{r}_i/m_p) = \sum_j r_{ij} P(w_j)$$

So, I_* and I^* coincide with the mathematical expectation relative to p, and therefore any function v provides us the classical criterion in a probabilistic risk environment.

b) Ignorance.

If no information at all is available about the nature states, this situation can be represented by a basic probability assignment m_o such that $m_o(\Omega) = 1$, and we have the decision problem in an uncertainty environment.

In this case, upper and lower integrals are:

$$I_*(\tilde{r}_i/m_o) = \operatorname{Min}_j r_{ij}$$

$$I^*(\tilde{r}_i/m_o) = \operatorname{Max}_j r_{ij}$$

So, we have:

$$v^*(g(\tilde{r}_i)) = \operatorname{Max}_j r_{ij}$$

$$v_*(g(\tilde{r}_i)) = \operatorname{Min}_j r_{ij}$$

$$v_\alpha(g(\tilde{r}_i)) = \alpha(\operatorname{Max}_j r_{ij}) + (1-\alpha)(\operatorname{Min}_j r_{ij})$$

The classical max-min and max-max criterions are obtained from v_* and v^*, respectively, and v_α provides Hurwicz criterion.

5.CONCLUSION.

In this paper, some rules to make decisions, under an information represented by an evidence, have been analysed. In the case, we have a probability as -- information (risk environment), the classical mathematical expectation criterion is obtained.

When the information is less precise, decisor's attitude is more important. In the extreme case of uncertainty environment an optimistic (max-max), pesimistic (max-min) or compensative (Hurwicz) criterion can be used.

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