MULTI-VALUED RELATIONS

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1. Preface

Some properties of real phenomenon are easily expressed in terms of numbers, some other, so called quantitative properties, are usually expressed by means of relations.

From classical point of view, binary relation in X is defined as a subset of the Cartesian product $X \times X$, or equivalently as a function χ_R defined on $X \times X$ and taking values on two element set $\{0,1\}$, and therefore sometimes is called two-valued relation.

In order to express some particular properties such as orderings, similarities, dissimilarities etc., some conditions which should be satisfied by corresponding relations are defined.
In practical applications these conditions are often violated.
This can be interpreted in two ways:

- we are either acting irrationaly, because we violate the conditions which are considered as axioms of rationality,
- or the conditions are improper or are wrong.

In the first case we must improve ourselvs saying: oh, sorry, I've made a mistake, in the latter however, we must change the theory i.e. we should improve the conditions according to the principle: theory for men but not vice-versa.

It was discovered much earlier, that black and white principle used in classical definitions is too rigid and should be weakened admitting shades of grey.

Verbally speaking it means that instead of insisting that statement "x is in relation with y" is true or false, it would be better to admit some grades of truthness.

Formaly it can be realized by repleacing function $\chi_R : X \times X \rightarrow R^1$ by function $\varphi_R : X \times X \rightarrow L$, where L is some prespecified set of valuations, and father on X will be assumed countable at most.

2. Definitions

In order to choose the proper set of valuations, it is desired that L is in certain sense generalization of two element set $\{0,1\}$. This means that L should contain $\{0,1\}$ and should have the same mathematical structure as $\{0,1\}$.

Two element set $\{0,1\}$ can be considered in a number of ways. It can be treated as partially ordered set, as lineary ordered set, or as some algebraic structure such as Boolean algebra, as monoid, as semigroup with unity, as semiring, as complete distributive lattice ect.

So that there are many possibilities to choose the structure for the set L. Which one should be chosen? It depends on practical needs.

First of all we need in practice to compare the grades of truthness, therefore L should be equipped with some order relation. Secondly, if we want to treat these grades not only "ordinaly" as some labels, but also as cardinal numbers, this would imply some operation on grades. So that L should be at least an ordered monoid. For many reasons it is required that operation is associative, it means that L should be an ordered semigroup.

The most exploited structure is an ordered semigroup defined on unit interval $[0,1]$. The reason is that people are used to fractions interpreted as probability, truth, grades of membership, credibility etc.

On the other hand, and what is more important, such a structure is deeply investigated by mathematicians and obtaind results are very attractive for applications [2,3].

One of such results is, for example, general formula "producing" semigroup operations $*$ on unit interval. This formula is:

$$a * b = f^{-1}(f(a) + f(b)) , \tag{1}$$

where f is continuous and strictly monotone function, and f^{-1} is the inverse of f.

The above semigroup is called Aczelian semigroup /see [6] /.

Some additional requarenments on $*$ yield Archimedian semigroup / [15] /:

$$a * b = f^{(-1)}(f(a) + f(b)) \quad a,b \in [0,1] \tag{2}$$

where $f^{(-1)}$ is a so called pseudo-inverse of f defined by:

$$f^{(-1)}(y) = \begin{cases} f^{-1}(y), & \text{if } y \in [0,\ f(0)] \\ 0 & , \text{otherwise} . \end{cases}$$

Function f is called the generator of operation $*$, if f is decreasing with $f(1) = 0$, then the operation $*$ will be called addition. If f is increasing with $f(0) = 0$, then $*$ will be called multiplication.

Let us assume now that L is the following ordered semigroup with unity $L = ([0,1], *, \leq, \varepsilon)$, where operation $*$ is defined by (2), $\leq$ is an order conciding with $*$

$$a \leq b \Rightarrow a * c \leq b * c,$$

and ε is a neutral element.

Valued relation R is defined by

$$R : X \times X \rightarrow L, \tag{3}$$

which is also called multi-valued relation, weghted relation, fuzzy relation, metric relation etc. / see [5,7,9] /.

In order to define some classes of relations we need to define the properties of relations in terms of function (3).

For this end let us introduce some necessery definitions.

Let $*^{-1}$ denotes quasi-inverse operation defined as follows:

$$b *^{-1} a = \inf \{x \in [0,1] \mid a * x \geq b\}, \text{ if } * \text{ is addition,} \tag{4}$$

$$b *^{-1} a = \sup \{x \in [0,1] \mid a * x \leq b\}, \text{ if } * \text{ is multiplication.} \tag{5}$$

Using only one operation, namely multiplication, let us introduce the following definitions:

$$\bar{R}(x,y) = 0 *^{-1} R(x,y) \tag{6}$$

$$R^{-1}(x,y) = R(y,x) \tag{7}$$

$$I(x,y) = R(x,y) * R(y,x) \tag{8}$$

$$P(x,y) = R(x,y), \text{ if } R(x,y) \geq R(y,x), \text{ otherwise } P(x,y) = 0 \tag{9}$$

$$R^n(x,y) = R^{n-1}(x,z) * R(z,y), \; n \geq 2 \tag{10}$$

From many possible definitions of properties of relation those which are gathered in the following table are choosen.

name of property	definition
P1. Reflexivity	$R(x,x) \geq l$, for some fixed $l \in L$
P2. Irreflexivity	$R(x,x) < l$
P3. Symmetry	$R(x,y) = R(y,x)$
P4. Asymmetry	$R(x,y) \neq R(y,x)$
P5. Antisymmetry	$R(x,y) = R(y,x) \Rightarrow x=y$
P6. Transitivity	$R(x,y) * R(y,z) \leq R(x,z)$
P7. Negative transitivity	$\bar{R}(x,y) * \bar{R}(y,z) \leq \bar{R}(x,z)$
P8. Connectedness	$R(x,y) \geq l \Rightarrow l \geq R(y,x)$
P9. Weak connectedness	$x \neq y \Rightarrow$ P8.
P10. Interval transitivity	$P(x,y) * I(y,w) * P(w,z) \leq P(x,z)$
P11. Semi transitivity	$P(x,y) * I(y,z) * I(z,w) \leq P(x,w)$
P12. Quasi transitivity	$P(x,y) * P(y,z) \leq P(x,z)$
P13. Strong transitivity	P12 & $x \neq y \Rightarrow [P(x,y) \geq l \Rightarrow l \geq P(y,x)]$
P14. Blau's gener. trans.	$P^s(x,y) * I(y,w) * P^t(w,z) \leq P(x,z)$

The main classes of relations are the following /see [4] /:

C1. preorder	P1, P6
C2. strct partial order	P2, P6
C3. interval order	P2, P10
C4. semi order	P2, P10, P11
C5. weak order	P4, P7
C6. linear order	P2, P6, P9 .

3. Criterial representation

Since functio $f : X \to R^1$ can have some natural empirical interpretation e.g. f(x) could be treated as an utility of x, value of x, goodness of x etc., therefore it is interesting to express function R in terms of functions belonging to some family $F = \{f_1, \ldots, f_n\}$. Family F could be treated as a set of criteria $f_1, \ldots, f_n$ which form the basis of relation R.

The problem of representation of multi-place function by composition or super osition of "simpler" functions, especially functions of fewer places is a very old one.
As early as 1934 W. Sierpiński has showed that evry two-place function g can be represented in the form /see [6] /:

$$g(x,y) = h(f_1(x) + f_2(y))$$

for some functions h, f_1, f_2.

This problem has been deeply investigated for some particular cases of functions g, mainly for continuous and associative functions. This problem was solved by J. Aczel in 1948.

In our case, two-place function R which we wanted to compose from one-place functions, is not only continuous and associative but it also satisfies some additional conditions listed in the previous paragraph.

The problem of representation of such functions has been investigated for the first time by E. Trillas, L. Valverde and S.V. Ovchinnikov /see [9,14]/.

Suppose that T means t-norm defined on $[0,1]$ such that

$$T(a,b) \geq a * b \ , \text{ for any } a,b \in [0,1] .$$

Relation R i.e. function (9) is a preorder if and only if there exists some family $F = \{f_1,\dots,f_n\}$ such that

$$R(x,y) = \mathop{T}_{i=1}^{n} \left(f_i(y) *^{-1} f_i(x) \right) \tag{11}$$

$$\text{where} \quad \mathop{T}_{i=1}^{n} a_i = T(a_1 T \dots T(a_{n-1}, a_n))$$

and it is assumed that $R(x,x) = 1$ i.e. $1 = 1$.

Let us consider some particular cases of this representation.

Case 1. Assume that T is inf, and $*$ is a usual multiplication, then we have [9]:

$$R(x,y) = \inf_i \left\{ \min \left(f_i(y)/f_i(x), \ 1 \right) \right\} . \tag{12}$$

Case 2. Considering separately the particular pairs (x,x) we obtain the representation of strict partial order O2:

$$R(x,y) = \inf_i \left\{ \min \left(f_i(y)/f_i(x), \ 1 \right) \right\} , \text{for } x \neq y, \tag{13}$$

$$R(x,x) = 0.$$

Case 3. If the family F consists only of two functions f_1 and f_2 such that $f_1(x) \leq f_2(x)$ for all $x \in X$, then we have the representation of interval order O3:

$$R(x,y) = \min \left\{ f_1(x)/f_2(y), \ 1 \right\} . \tag{14}$$

Case 4. Relation R is a semiorder O4 if and only if there exists family $F = \{f_1, f_2\}$ such that $f_2(x) = c = \text{const}$ and

$$R(x,y) = \min \left\{ f_1(x)/(f_1(y) + c), \ 1 \right\} . \tag{15}$$

Case 5. Relation R satisfies conditions P4 and P7 i.e. R is weak order O5 if and only if F contains only one function $f : X \to R^1$ such that

$$R(x,y) = f(y) *^{-1} f(x) . \tag{16}$$

Case 6. Relation R is an linear order if and only if function $f : X \to R^1$ fulfills the additional condition $x \neq y \Rightarrow f(x) \neq f(y)$ and satisfies

$$R(x,y) = f(y) *^{-1} f(x). \tag{17}$$

4. Intuitive explanation

It is well known that function $f : X \to R^1$ induce a linear order $R \subseteq X \times X$ by

$$(x,y) \in R \Leftrightarrow f(x) > f(y)$$

and conversaly: for any linear order there is a function which define it. More formally: relation R fulfills properties

$$x \bar{R} x \tag{18}$$

$$x R y \Rightarrow y \bar{R} x \tag{19}$$

$$x R y \,\&\, y R z \Rightarrow x R z \tag{20}$$

if and only if there is function $f : X \to R^1$ such that

$$x R y \Leftrightarrow f(x) > f(y). \tag{21}$$

Let us rewrite (21) in the two f o r m a l l y equivalent forms:

$$x R y \Leftrightarrow \alpha(x,y) > 0, \tag{22}$$

$$x R y \Leftrightarrow \mu(x,y) > 1, \tag{23}$$

where $\alpha(x,y) = f(x) - f(y)$ and $\mu(x,y) = f(x)/f(y)$.

It means that relation $R \subseteq X \times X$ can be defined by some two-place function $\varphi_R : X \times X \to R^1$ according to the scheme $x R y \Leftrightarrow \varphi_R(x,y) > 0$.

Let R be read as "better", then $\varphi_R(x,y) = \alpha(x,y)$ could be interpreted as "x is better by $\alpha(x,y)$ than y", and in the case when $\varphi_R(x,y) = \mu(x,y)$ we have the interpretation: "x is better $\mu(x,y)$ times than y".

Providing that such an interpretation is justified, relation $R \subseteq X \times X$ can be identified with function $\varphi_R : X \times X \to R^1$. For the sake of simplicity, symbol R will be used instead of φ_R.

The problem is now to reformulate the properties (18), (19) and (20). Let us take the following reformulation:

$$R(x,x) = 0 \tag{24}$$

$$R(x,y) + R(y,x) = 0 \tag{25}$$

$$R(x,y) + R(y,z) = R(x,z) \tag{26}$$

We have then the following result: R fulfiells (24), (25) and (26) if and only if there exists function $f : X \to R^1$ such that

$$R(x,y) = f(x) - f(y). \tag{27}$$

Let us take other reformulation now:

$$R(x,x) = 1, \tag{28}$$

$$R(x,y) \cdot R(y,x) = 1, \tag{29}$$

$$R(x,y) \cdot R(y,z) = R(x,z), \tag{30}$$

then instead of (27) we obtain

$$R(x,y) = f(x) / f(y). \tag{31}$$

So that (27) and (31) are both of exactly the same form as (17). Which representation should be or can be chosen ?

The answer is simple: this one which is meaningful.

The notion of meaningfulness in the theory of measurement is understood as follows /see [10,11]/: a statement involving numerical scale is meaningful if its truth /or falsity/ remains unchanged if every scale involved is repleaced by another acceptable scale.

In our case we reverse the problem: the representation (27) involvs the defference $f(x) - f(y)$, we are looking for admissible transformation which preservs this difference /see also [8]/. It turns out that these transformations are of the form $\phi(x) = x+c$. This means that representation (27) is meaningful only for difference scale. Similarly, representation (31) is meaningful only if f is an ratio scale.

Suppose now that instead of $\varphi_R : X \times X \to R^1$ we are looking for representation in the form of $\varphi_R : X \times X \to [0, \infty)$, Since sub traction is not performable in $[0, \infty)$, then we define quasi-subtraction

$$b \dot{-} a = \begin{cases} b - a, & \text{if } b > a, \\ 0, & \text{otherwise,} \end{cases}$$

or in an equivalent form

$$b \dot{-} a = \max \{b - a, 0\}.$$

Relation R fulfills (24), (25) and (26) if and only if there is $f : X \to R^1$ such that

$$R(x,y) = f(y) \dot{-} f(x) . \qquad (32)$$

This is again of the form of (17).

For the multiplicative case we obtain the similar result. Namely, $R : X \times X \to [0,1]$ fulfills (28), (29) and (30) if and only if there exists $f : X \to R^1$ such that

$$R(x,y) = f(y) \dot{\div} f(x), \qquad (33)$$

where $\qquad b \dot{\div} a = \min \{b/a, 1\}$.

The meaningfulness of (32) and (33) requires also a difference and ratio scales respectively.

Let us suppose that we want to treat fractions obtained by (33) as grades of membership defining some fuzzy relation. If we want to consider them as cardinal numbers, then it is easy to see that for being meaningful there is needed at least ratio scale. Otherwise grades of membership can be treated only as measurements in ordinal scale.

For example, there is no meaningful binary fuzzy relation "warmer", unless the grades are treated as some labeles but not numbers. For the same reason fuzzy relation "greater" without empirical interpretation remains only as a mathematical curiosity.

5. Example

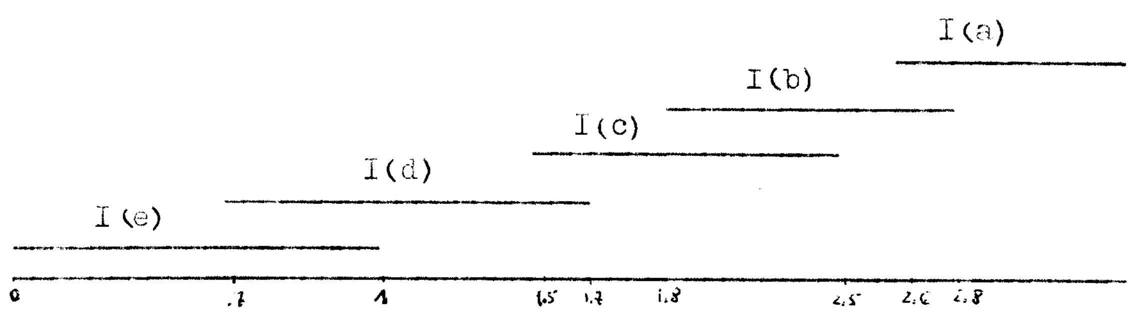
Let $X = \{a,b,c,d,e\}$. Suppose there are given criteria $f_1, f_2, f_3 : X \to R^1$ defined as follows:

	a	b	c	d	e
f_1	3.6	1.3	2.5	.7	1
f_2	2.6	2.5	1.5	1.7	1
f_3	2.8	2.6	1.7	1.5	1

According to (13) the following strict partial order is induced:

$$R = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & .93 & 1 & 1 & 1 \\ & 0 & .72 & 1 & 1 \\ & & 0 & .88 & 1 \\ & & & 0 & .7 \\ & & & & 0 \end{bmatrix} \end{array}$$

It is not difficult to check that relation R fulfills properties P2 and P10 i.e. is an interval order. According to (14) this relation is determined by two functions, let us denote them now by f^- and f^+. It means that an interval $I(x) = [f^-(x), f^+(x)]$ such that $x\ R\ y \Leftrightarrow I(x) < I(y)$ can be assigned now to any $x \in X$. One of the possible interval assignments is presented below.



Interval order considered above satisfies also property P11 i.e. as a matter of fact it is semiorder, therefore representation (15) is possible. Indeed, let $f(a)=2.6$, $f(b)=1.8$, $f(c)=1.5$, $f(d)=.7$, $f(e)=0$, and let $c=1$, then $R(x,y) = \min\{f(x)/(f(y)+1),\ 1\}$.

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