FUZZY RELATIONAL NON-DETERMINISTIC EQUATION AND SOLUTION OF THE SCHIN RANK OF A FUZZY MATRIX

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ABSTRACT

In this paper we present a definition of a fuzzy relational non-deterministic equation, discusse preliminarily its properties, and give a convenient algorithm for detdrmieing the Schein rank by using a fuzzy relational non-deterministic equation.

Keyword: Fuzzy Relational Non-Deterministic Equation.

I. FUZZY RELATIONAL NON-DETERMINISTIC EQUATION

A definition of fuzzy relational equation is given in[1]. Here, we shall define fuzzy relational non-deterministic equation.

By convention, we write $a+b=\max\{a,b\}$, $ab=\min\{a,b\}$, Where $a,b\in[0,1]$.

The sum of two fuzzy matrices ($A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{m\times n}$) is written as $A+B=(a_{ij}+b_{ij})_{m\times n}$.

The scalar product of a number and a fuzzy matrix is written as $A=(ka_{ij})_{m\times n}=(\min\{k,a_{ij}\})$, $k\in[0,1]$.

We write the product of two fuzzy matrices ($A=(a_{ij})_{m\times t}$, $B=(b_{ij})_{t\times n}$) as $AB=(c_{ij})_{m\times n}$, $c_{ij}=\sum_{k=1}^{t} a_{ik}b_{kj}$.

Definition 1.1 Let $A=(a_{ij})_{m\times n}$, and A be a non-zero fuzzy matrix. A equation in the form of

$$A_{m\times n}=Y_{m\times s}X_{s\times n} \qquad \ldots\ldots(1)$$

is called a fuzzy reletional non-deterministic equation of the fuzzy matrix A or a non-deterministic equation of A. (where A is known, Y and X are unknown). s is called an index. Y and X such that (1) holds are called the solution matrices of the non-deterministic equation of A of index s.

The following propertice on a fuzzy relational non-deterministic equation are easily proved:

Theorem 1.1 The non-deterministic equation of a m×n fuzzy matrix A has solutions if s=n or s=m.

Theorem 1.2 For an index $s=s_0$, if the non-deterministic equation of a m×n fuzzy matrix A has solutions, then the non-deterministic equation of A also has solution if $s=s_0+k$, where k is positive indeger.

Theorem 1.3 The non-deterministic equation of a m×n fuzzy matrix $A=(a_{ij})_{m\times n}$ always solutions if the index $s \geq \min\{m,n\}$.

Theorem 1.4 The non-deterministic equation of a m×n fuzzy matrix $A=(a_{ij})_{m\times n}$ has solutions while $s=s_0$ and $Y_{m\times s_0}, X_{s_0\times n}$ are its solution, and $Y_{m\times s_0}$, $Z_{s_0\times n}$ are also its solution, then $Y_{m\times s_0}$, $(Z_{s_0\times n}+X_{s_0\times n})$ also are its solution.

Theorem 1.4' The non-deterministic equation of a m×n fuzzy matrix $A=(a_{ij})_{m\times n}$ has a solution while if $s=s_0$, $Y_{m\times s_0}$, $X_{s_0\times n}$ are a solution, and $Z_{m\times s_0}, X_{s_0\times n}$ are also its solution, then

$(Z_{m\times s_0}+Y_{m\times s_0})$, $X_{s_0\times n}$ are also its solution.

Theorem 1.5 The non-deterministic equation of a m×n fuzzy matrix $A=(a_{ij})_{m\times n}$ while $s=s_0$ has a solution and $Y_{m\times s_0}$, $X_{s_0\times n}$ are its solution. Let $Y_{m\times s_0}=(y_{ij})_{m\times s_0}$, and $k=\max_{i,j}\{y_{ij}\}$, then for $1 \geq h \geq k$, $hY_{m\times s_0}$, $X_{s_0\times n}$ are also its solution and $Y_{m\times s_0}$, $hX_{s_0\times n}$ are also its solution.

Theorem 1.5' The non-deterministic equation of a m×n fuzzy matrix $A=(a_{ij})_{m\times n}$ while $s=s_0$ has a solution and $Y_{m\times s_0}$, $X_{s_0\times n}$ are its solution. Let $X_{s_0\times n}=(x_{ij})_{s_0\times n}$, $k=\max_{i,j}\{x_{ij}\}$. Then for $1 \geq h \geq k$, $Y_{m\times s_0}$, $hX_{s_0\times n}$ are its solution, $hY_{m\times s_0}$, $X_{s_0\times n}$ are also its solution.

Theorem 1.6 The non-deterministic equation of a m×n fuzzy matrix $A=(a_{ij})_{m\times n}$, while $s=s_0$ has a solution, and $Y_{m\times s_0}$, $X_{s_0\times n}$ are its solution. Let $k=\max_{i,j}\{a_{ij}\}$, then for $1 \geqslant h \geqslant k$, $hY_{m\times s_0}$, $X_{s_0\times n}$ are its solution, and $Y_{m\times s_0}$, $hX_{s_0\times n}$ are also its solution.

Theorem 1.7 The non-deterministic equation of a mxn fuzzy matrix $A=(a_{ij})_{m\times n}$ has a solution while s=1 if and only if A can be expressed as such form:

$$A=\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}(b_1,\dots,b_n).$$

From the theorem and theorem 1 in (4), we concludes:

Theorem 1.8 The non-deterministic equation of a mxn fuzzy matrix A for index s=1 has a solution if and only if rows and columns of A and its submatrices are crossed out step by step up to 1X1 submatrix, and A can be given by a product of two vectors of the column vetor and row vector containing the maximum element of A.

Explanation: If some row (or column) of a matrix A has identical elements , then this row (or column) is called the same element row (or column) of A.

If the minimal element of a matrix A forms a row (or column) of A , then are say that the row (or column) can be crossed cut , and other rows (columns) form a new matrix, which is called a submatrix of A.

For example: Let

$$A=\begin{bmatrix} 0.6 & 0.7 & 0.6 & 0.5 & 0.2 & 0.1 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.2 & 0.1 \\ 0.8 & 0.6 & 0.7 & 0.5 & 0.2 & 0.1 \end{bmatrix}$$

then cross out step by step as follows:

$$\begin{bmatrix} 0.6 & 0.7 & 0.6 & 0.5 & 0.2 & 0.1 \\ 0.4\cdots & 0.4\cdots & 0.4\cdots & 0.4\cdots & 0.2\cdots & 0.1 \\ 0.8 & 0.6 & 0.7 & 0.5 & 0.2 & 0.1 \end{bmatrix}\cdots 3$$
$$\qquad\qquad\qquad\qquad\qquad\qquad 4 \qquad 2 \qquad 1$$

so get submatrix:

$$\begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$$

Some algorithms of solving the non-deterministic equation of a mxn fuzzy matrix A,specially flow chart of solving a fuzzy relational non-deterministic equation and the program in FORTRAN are given in (5), here we omit the program.

II. THE ALGORITHM OF SOLVING THE SCHIN RANK OF A MATRIX A

Definition 2.1 (2) The Schin rank $\rho_s(A)$ of a matrix A is the least number of rank 1 matrices shose sum is A.

Proposition 2.1 (2) $\rho_s(A) \leqslant \min\{m,n\}$ for a fuzzy matrix $A_{m\times n}$.

Proposition 2.2 (i) If a matrix A only has one row (or one column) then $\rho_s(A)=1$. (ii) $\rho_s(A)=1$ if and only if for A and its submatrices can be crossed out rows and columns step by step up to a 1×1 submatrix.

Further we have:

Theorem 2.1 For a mxn fuzzy matrix A $\rho_s(A)=s$ if and only if the non-deterministic equation ofAhas no solution while $1 \leqslant t \leqslant s-1$, but has a solution while s=t.

Proof. Necessity. Since $\rho_s(A)=s$, there exist

$$(y_{11}^o,\ldots,y_{m1}^o)^T,\ldots,(y_{1s}^o,\ldots,y_{ms}^o)^T$$

and $(x_{11}^o,\ldots,x_{1n}^o),\ldots,(x_{s1}^o,\ldots,x_{sn}^o)$ such that

$$A=\begin{bmatrix} y_{11}^o \\ \vdots \\ y_{m1}^o \end{bmatrix}(x_{11}^o,\ldots,x_{1n}^o)+\ldots+\begin{bmatrix} y_{1s}^o \\ \vdots \\ y_{ms}^o \end{bmatrix}(x_{s1}^o,\ldots,x_{sn}^o)$$

$$=\begin{bmatrix} y_{11}^o,\ldots,y_{1s}^o \\ \vdots \qquad \vdots \\ y_{m1}^o,\ldots,y_{ms}^o \end{bmatrix}\begin{bmatrix} x_{11}^o,\ldots,x_{1n}^o \\ \vdots \qquad \vdots \\ x_{s1}^o,\ldots,x_{sn}^o \end{bmatrix} = Y_{m\times s}^o \; X_{s\times n}^o$$

where

$$Y^{o}_{m\times s}=\begin{bmatrix} y^{o}_{11},\dots,y^{o}_{1s} \\ \vdots \qquad \vdots \\ y^{o}_{m1},\dots,y^{o}_{ms}\end{bmatrix} \quad \text{and} \quad X^{o}_{s\times n}=\begin{bmatrix} x^{o}_{11},\dots,x^{o}_{1n} \\ \vdots \qquad \vdots \\ x^{o}_{s1},\dots,x^{o}_{s\,n}\end{bmatrix},$$

Thus $Y^{o}_{m\times s}$ and $X^{o}_{s\times n}$ are solutions of the non-deterministic equation of A for t=s. Then , if t=s, the equation has a solution.

For $1 \leq t \leq s-1$, the non-deterministic equation of A has no solution. If it is not true, we suppose that the non-deterministic equation of A has a solution for some index, where $1 \leq t_1 \leq s-1$. Let its one solution as follows:

$$Y^{1}_{m\times t_1}=\begin{bmatrix} y^{1}_{11},\dots,y^{1}_{1t_1} \\ \vdots \qquad \vdots \\ y^{1}_{m1},\dots,y^{1}_{mt_1}\end{bmatrix} \qquad X^{1}_{t_1\times n}=\begin{bmatrix} x^{1}_{11},\dots,x^{1}_{1n} \\ \vdots \qquad \vdots \\ x^{1}_{t_1 1},\dots,x^{1}_{t_1 n}\end{bmatrix}$$

then

$$A=Y^{1}_{m\times t_1}X^{1}_{t_1\times n}=\begin{bmatrix} y^{1}_{11},\dots,y^{1}_{1t_1} \\ \vdots \qquad \vdots \\ y^{1}_{m1},\dots,y^{1}_{mt_1}\end{bmatrix} \cdot \begin{bmatrix} x^{1}_{11},\dots,x^{1}_{1n} \\ \vdots \qquad \vdots \\ x^{1}_{t_1 1},\dots,x^{1}_{t_1 n}\end{bmatrix}$$

$$=\begin{bmatrix} y^{1}_{11} \\ \vdots \\ y^{1}_{1n}\end{bmatrix}(x^{1}_{11},\dots,x^{1}_{1n})+\dots+\begin{bmatrix} y^{1}_{1t_1} \\ \vdots \\ y^{1}_{mt_1}\end{bmatrix}(x^{1}_{t_1 1},\dots,x^{1}_{t_1 n})$$

so $\rho_s(A) \leqslant t_1 \leqslant s-1$. But $\rho_s(A)=s$ which is a contradiction. It is easy to prove the sufficiency.

From the theorem, we know that, the problem of finding the Schein rank of a fuzzy matrix A can be turn into a problem of finding minimum index , provided the non-deterministic equation of this matrix A has a solution.

Example. Let

$$A=\begin{bmatrix}0.4 & 0.4 & 0.9\\0.3 & 0.7 & 0.3\\0.4 & 0.2 & 0.8\\0.5 & 0.8 & 0.5\end{bmatrix}$$

Are have programme a FORTRAN program corresponding to the flow chart in (5). Using Z80 computer to compute the problem are know that the non-deterministic equation of A has no solution With if its index is equal to 2, and A is a 3-column matrix, thus $\rho_s(A)= 3$. (In(4), an estimation of Schin rank of the matrix A is give merely).

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