ON FUZZY SEMI-ALGEBRAS

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The families of fuzzy measurable subsets in X are usually defined as follows:

Definition 1: Each family $\sigma \subset \mathbb{F}(X)$ having the properties:

$$\mathbb{0}_X \in \sigma \quad , \tag{1}$$

$$\forall_{\mu \in \sigma} \qquad 1 - \mu \in \sigma \quad , \tag{2}$$

$$\forall_{(\mu, \nu) \in \sigma^2} \qquad \mu \vee \nu \in \sigma \tag{3}$$

is called a fuzzy algebra [1].

Definition 2: Each fuzzy algebra σ fulfilling additionally the condition

$$\forall_{\{\mu_n\} \in \sigma^{\mathbb{N}}} \qquad \sup_n \{\mu_n\} \in \sigma \tag{4}$$

is called a fuzzy σ-algebra [1].

Attachment of the empty set $\mathbb{0}_X$ and the universum $\mathbb{1}_X$ (it follows from (1) and (2)) is necessary not only for mathematical considerations but also for practical applications. The empty set and the universum can be defined more general than above. The following qualifications are proposed:

Definition 3: Each $\mu \in \mathbb{F}(X)$ satisfying $\mu \leqslant 1 - \mu$ is called a W-empty set [5].

Definition 4: Each $\mu \in \mathbb{F}(X)$ satisfying $\mu \geqslant 1 - \mu$ is called a W-universum [5].

For this case we have:

Theorem 1: Each $\mu \in \mathbb{F}(X)$ is a W-empty set iff there exists such $\nu \in \mathbb{F}(X)$ that $\mu = \nu \wedge (1 - \nu)$ [5].

Theorem 2: Each $\mu \in \mathbb{F}(X)$ is a W-universum iff there exists such $\nu \in \mathbb{F}(X)$ that $\mu = \nu \vee (1 - \nu)$ [5].

Consider now the following definitions:

Definition 5: Each nonempty family $\mathfrak{S} \subset \mathbb{F}(X)$ satisfying the conditions (2) and (3) is called a fuzzy semi-algebra.

Definition 6: Each fuzzy semi-algebra fulfilling the condition (4) is called a fuzzy $\mathfrak{S}$-semi-algebra.

Theorem 3: Each fuzzy semi-algebra ($\mathfrak{S}$-semi-algebra) $\mathfrak{S}$ contains a W-empty set and a W-universum.

Proof: Let $\mu \in \mathfrak{S}$. Taking (2) and (3), we get $\mu \vee (1 - \mu) \in \mathfrak{S}$ and $\mu \wedge (1 - \mu) \in \mathfrak{S}$. In agreement with the Theorems 1 and 2, $\mathfrak{S}$ contains a W-empty set and a W-universum. ■

So, from practical point-view, fuzzy algebras can be replaced by fuzzy semi-algebras.

Take into account the fuzzy subset $[\![\frac{1}{2}]\!]_X : X \to \{\frac{1}{2}\}$. We observe that:

- $[\![\frac{1}{2}]\!]_X$ is both W-empty set and W-universum;
- $[\![\frac{1}{2}]\!]_X$ is an unmeasurable fuzzy subset in sense given in [2];
- $[\![\frac{1}{2}]\!]_{X\times X}$ presents such fuzzy relation on X that it is both reflexive, antireflexive, symmetrical and transitive in sense given in [3].

Therefore, I am proposing to accept the following qualifications:

Definition 7: Each fuzzy semi-algebra (σ-semi-algebra) uncontaining the fuzzy subset $[\![\frac{1}{2}]\!]_X$ is called a soft fuzzy semi-algebra (σ-semi-algebra).

Considering soft fuzzy semi-algebras only, we can avoid mentioned above difficulties.

In [6], a families of fuzzy measurable subsets in the real line $\bar{\mathbb{R}} = [-\infty, +\infty]$ were described as fuzzy algebras, $\widehat{\beta}_{\leq}$ and $\beta_{\leq}$ say. These algebras were generated by such quasi-antisymmetrical and continuous from above fuzzy relation "less or equal" that it unfuzzily bounds the real line (see [4]).

Let us consider now such families of fuzzy subsets in $\bar{\mathbb{R}}$, similar to $\widehat{\beta}_{\leq}$ and $\beta_{\leq}$, which are generated by quasi-antisymmetrical fuzzy relation "less or equal" defined as follows:

Definition 8: Each mapping $\varrho : \bar{\mathbb{R}}^2 \to [0,1]$ fulfilling the next conditions

$$\varrho(x,y) \geqslant 1 - \varrho(y,x) \quad , \tag{5}$$

$$\varrho(x,y) + \varrho(z,x) \leqslant 1 \quad , \tag{6}$$

$$\varrho(z,y) < \frac{1}{2} \tag{7}$$

for each $(x,y,z) \in \bar{\mathbb{R}}^3$ with $y < z$, is called a quasi-antisymmetrical fuzzy relation "less or equal" (FLE) [4].

Definition 9: Each mapping $\varrho_s : \bar{\mathbb{R}}^2 \to [0,1]$, given by

$$\forall\,(x,y) \in \bar{\mathbb{R}}^2 \qquad \varrho_s(x,y) = 1 - \varrho(y,x) \quad , \tag{8}$$

is called a fuzzy relation "less than" (FLT) generated by FLE [4].

Lemma 1: Any quasi-antisymmetrical FLE ϱ and generated by it FLT ϱ_s satisfy the next inequalities:

$$\varrho(x,x) \geqslant \frac{1}{2} > \varrho(y,x) \geqslant \varrho_s(y,x) \quad , \tag{9}$$

$$\varrho(x,y) \geqslant \varrho_s(x,y) > \frac{1}{2} \geqslant \varrho_s(x,x) \quad , \tag{10}$$

$$\varrho(\cdot,x) \leqslant \varrho(\cdot,y) \quad , \tag{11}$$

$$\varrho(y,\cdot) \leqslant \varrho(x,\cdot) \quad , \tag{12}$$

$$\varrho_s(\cdot,x) \leqslant \varrho_s(\cdot,y) \quad , \tag{13}$$

$$\varrho_s(y,\cdot) \leqslant \varrho_s(x,\cdot) \tag{14}$$

for each $(x,y) \in \bar{\mathbb{R}}^2$ such that $x < y$ [4].

Using the fixed quasi-antisymmetrical FLE ϱ and generated by it FLT ϱ_s we can define a fuzzy intervals as mappings $\varphi[a,b[: \bar{\mathbb{R}} \to [0,1]$ and $\varphi[a,+\infty] : \bar{\mathbb{R}} \to [0,1]$, given by means of the identities:

$$\varphi[a,b[\,(x) = \varrho(a,x)\wedge\varrho_s(x,b) =$$
$$= (1-\varrho_s(x,a))\wedge\varrho_s(x,b)\ , \tag{15}$$
$$\varphi[a,+\infty]\,(x) = \varrho(a,x)\wedge\varrho(x,+\infty) \tag{16}$$

for each $(a,b,x)\in\overline{\mathbb{R}}^3$. Among others things, for intervals defined above, we have:

Lemma 2: The conditions:

$$\varphi[b,c[\,\wedge\varphi[-\infty,a[\, = \varphi[b,a\wedge c[\ , \tag{17}$$
$$\varphi[-\infty,a[\, = \varrho_s(\cdot,a)\ , \tag{18}$$
$$\varphi[a,+\infty] = \varrho(a,\cdot) \tag{19}$$

hold for any triplet $(a,b,c)\in\overline{\mathbb{R}}^3$.

Proof: The identity (17) is shown in [6] . Moreover, the inequalities (5), (9), (10), (11) and (12) imply:

- for $-\infty\leqslant x<a$

  $\varrho(-\infty,x)\geqslant\frac{1}{2}>\varrho_s(x,a)$;
- for $-\infty\leqslant a\leqslant x$

  $\varrho(-\infty,x)\geqslant\varrho(-\infty,a)\geqslant\varrho(x,a)\geqslant\varrho_s(x,a)$.

So, $\varrho(-\infty,\cdot)\geqslant\varrho_s(\cdot,a)$ for each $a\in\overline{\mathbb{R}}$. Puting this result in (15), we get (18). By analogous way as above, we obtain:

- for $+\infty\geqslant x>a$

  $\varrho(x,+\infty)\geqslant\frac{1}{2}>\varrho(a,x)$;
- for $+\infty\geqslant a\geqslant x$

  $\varrho(x,+\infty)\geqslant\varrho(a,+\infty)\geqslant\varrho(a,x)$.

Thus $\varrho(\cdot,+\infty)\geqslant\varrho(a,\cdot)$ for all $a\in\overline{\mathbb{R}}$. So, the condition (19) holds, too. ∎

In case when FLE also unfuzzily bounds $\bar{\mathbb{R}}$, the below definition describes a fuzzy algebra in $\bar{\mathbb{R}}$.

Definition 10: If $\hat{\beta}_{\varsigma} \subset \mathbb{F}(\bar{\mathbb{R}})$ is a family of such fuzzy subsets that

$$\mu = \mu_1 = \max_{k\leq n} \{\varphi[a_k, b_k[\} \tag{20}$$

or

$$\mu = \mu_2 = \max_{k\leq n} \{\varphi[a_k, b_k[\} \vee \varphi[a_{n+1}, +\infty] \tag{21}$$

where $\{a_k\}$ and $\{b_k\}$ are increasing sequences of numbers in $\bar{\mathbb{R}}$, then $\hat{\beta}_{\varsigma}$ is called a fuzzy finite Borel family [6] .

Let us look into the case when FLE is quasi-antisymmetrical only.

Lemma 3: Let be given any nondecreasing sequences $\{\mu_n\}$ and $\{\psi_n\}$. Then the sequence $\{\nu_n\}$ defined by $\nu_n = \mu_n \wedge (1-\psi_n)$ satisfies the identity

$$1 - \max_{k\leq n} \{\nu_k\} = \psi_1 \vee \max_{k<n} \{(1-\mu_k) \wedge \psi_{k+1}\} \vee (1-\mu_n) \quad . \tag{22}$$

Proof: For $n=1$, the identity (22) follows from the de Morgan's Law. Assume that (22) holds for each $n \leq m$. Then, for $n = m + 1$, we obtain

$$1 - \max_{k\leq m+1} \{\nu_k\} = 1 - \max_{k\leq m} \{\nu_k\} \vee \nu_{m+1} = (1 - \max_{k\leq m} \{\nu_k\}) \wedge (1 - \nu_{m+1}) =$$

$$= (\psi_1 \vee \max_{k<m} \{(1-\mu_k) \wedge \psi_{k+1}\} \vee (1-\mu_m)) \wedge ((1-\mu_{m+1}) \vee \psi_{m+1}) =$$

$$= (\psi_1 \wedge ((1-\mu_{m+1}) \vee \psi_{m+1})) \vee \max_{k<m} \{((1-\mu_k) \wedge \psi_{k+1}) \wedge$$

$$\wedge ((1-\mu_{m+1}) \vee \psi_{m+1})\} \vee ((1-\mu_m) \wedge ((1-\mu_{m+1}) \vee \psi_{m+1})) =$$

$$= \Psi_1 \vee \max_{k<m} \{((1-\mu_{m+1}) \wedge \Psi_{k+1}) \vee ((1-\mu_k) \wedge \Psi_{k+1})\} \vee$$

$$\vee ((1-\mu_m) \wedge \Psi_{m+1}) \vee (1-\mu_{m+1}) =$$

$$= \Psi_1 \vee \max_{k<m} \{(1-\mu_k) \wedge \Psi_{k+1}\} \vee ((1-\mu_m) \wedge \Psi_{m+1}) \vee (1-\mu_{m+1}) =$$

$$= \Psi_1 \vee \max_{k<m+1} \{(1-\mu_k) \wedge \Psi_{k+1}\} \vee (1-\mu_{m+1}) \quad . \blacksquare$$

Theorem 4: If FLE ϱ is quasi-antisymmetrical, then generated by it fuzzy finite Borel family $\widehat{\beta}_\varrho$ is a soft fuzzy semi-algebra. Furthermore, any fuzzy subsets μ_1 and μ_2, defined respectively by (20) and (21), satisfy:

$$1 - \mu_1 = \varphi[-\infty, a_1[\vee \max_{k<n} \{\varphi[b_k, a_{k+1}[\} \vee \varphi[b_n, +\infty] \quad , \tag{23}$$

$$1 - \mu_2 = \varphi[-\infty, a_1[\vee \max_{k\leq n} \{\varphi[b_k, a_{k+1}[\} \quad . \tag{24}$$

Proof: By (8), (15), (18), (19) and (22), we get

$$1 - \mu_1 = \varrho_s(\cdot, a_1) \vee \max_{k<n} \{(1 - \varrho_s(\cdot, b_k)) \wedge \varrho_s(\cdot, a_{k+1})\} \vee$$

$$\vee (1 - \varrho_s(\cdot, b_n)) = \varphi[-\infty, a_1[\vee \max_{k<n} \{\varphi[b_k, a_{k+1}[\} \vee \varphi[b_n, +\infty] \quad .$$

Nextly, by de Morgan's Law, (8), (17), (18), (19) and (23), we obtain

$$1 - \mu_2 = (1 - \max_{k\leq n} \{\varphi[a_k, b_k[\}) \wedge (1 - \varphi[a_{n+1}, +\infty]) =$$

$$= (\varphi[-\infty, a_1[\vee \max_{k<n} \{\varphi[b_k, a_{k+1}[\} \vee \varphi[b_n, +\infty]) \wedge \varphi[-\infty, a_{n+1}[\; =$$

$$= \varphi[-\infty, a_1 \wedge a_{n+1}[\vee \max_{k<n} \{\varphi[b_k, a_{k+1} \wedge a_{n+1}[\} \vee (\varrho(b_n, \cdot) \wedge$$

$$\wedge \varrho_s(\cdot, a_{n+1})) = \varphi[-\infty, a_1[\vee \max_{k\leq n} \{\varphi[b_k, a_{k+1}[\} \quad .$$

So, $\widehat{\beta}_\varrho$ is closed under complementation, It is self-evident that $\widehat{\beta}_\varrho$ is closed under union.

Since the set $\{x: x \in \bar{\mathbb{R}} \quad \mu(x) = \frac{1}{2}\}$ is a finite subset in $\bar{\mathbb{R}}$ for each $\mu \in \hat{\beta}_\varsigma$, $[\![\frac{1}{2}]\!]_{\bar{\mathbb{R}}} \notin \hat{\beta}_\varsigma$. ∎

Last of all, we define the following fuzzy σ-semi-algebra in $\bar{R}$.

Definition 11: The smallest fuzzy σ-semi-algebra β^*_ς containing $\hat{\beta}_\varsigma$ is called a fuzzy infinite Borel semi-family.

Theorem 5: Any fuzzy infinite Borel semi-family is a soft fuzzy σ-semi-algebra. Moreover, each $\mu \in \beta^*_\varsigma$ can be described by

$$\mu = \mu_1 = \sup_k \{\varphi[a_k, b_k[\} \tag{25}$$

or

$$\mu = \mu_2 = \sup_k \{\varphi[a_k, b_k[\} \vee \varphi[a_0, +\infty] \tag{26}$$

where $\{a_k\}$ $\{b_k\}$ are finite or infinite sequence of numbers in $\bar{\mathbb{R}}$.

Proof: Let the symbol Φ_ς denotes the family of all fuzzy subsets defined by (25) or (26). Additionally, let us indicate $\bar{\Phi}_\varsigma = \{\mu : 1 - \mu \in \Phi_\varsigma\}$.

From the Definition 10 we get that each $\mu \in \hat{\beta}_\varsigma$ can be described by (25) or (26). Since $\hat{\beta}_\varsigma$ is closed under complementation, we have $\hat{\beta}_\varsigma \subset \Phi_\varsigma \cap \bar{\Phi}_\varsigma$. This fact, along with the Definition 11, implies that $\beta^*_\varsigma \subset \Phi_\varsigma \cap \bar{\Phi}_\varsigma$, because $\Phi_\varsigma \cap \bar{\Phi}_\varsigma$ is a fuzzy σ-semi-algebra. By means of (4), we get $\Phi_\varsigma \subset \beta^*_\varsigma$. Taking into account all above facts, we see $\Phi_\varsigma \cap \bar{\Phi}_\varsigma \subset \Phi_\varsigma \subset \beta^*_\varsigma \subset \Phi_\varsigma \cap \bar{\Phi}_\varsigma$. So, $\beta^*_\varsigma = \Phi_\varsigma$. The second thesis is proved. Furthermore, the identities (25) and (26) imply that the set $\{x: x \in \bar{\mathbb{R}} , \mu(x) = \frac{1}{2}\}$ is numerous subset in

$\bar{\mathbb{R}}$ for each $\mu \in \mathcal{P}_{\varrho}^{*}$. Thus $[\![\frac{1}{2}]\!]_{\bar{\mathbb{R}}} \notin \mathcal{P}_{\varrho}^{*}$. ■

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