FUZZY RANDOM SET AND FUZZY SET-VALUED STATISTICS

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ABSTRACT

Wang peizhuang established the concept of the Set-Valued statistics in paper [1], its theoretical base is the theory of the random set and projected function. In this paper, we will extend these concepts and propose the method of fuzzy set-valued statistics. And the relation between the probabilistic set proposed by K.Hirota and fuzzy random set is discussed as well.

KEYWORDS: Fuzzy random set, Projected function, Fuzzy set-valued statistics, Probabilistic set, Degree of stability.

1.INTRODUCTION

The concept of set-valued statistics suggested by Wang peizhuang in paper [1] is a model of statistics in which each experimental trial gets a subset of sampling space, it is an extension of the classical probabilistic statistics and has extended the applicable range of the statistical experiment. Furthemore, the system of humanity, such as society, economics has suggested a new numercal method of management.

The theoretical base of set-valued statistics is the theory of the random set and projected function. In this respect, Prof. Wang peizhuang has done a lot of creative work[1]~[3]. On hypermeasurable constitution of pure measure, he has given a statement of pure measure form for random set.

On the process of set-valued statistical experiment, the observation result of every experiment is a subset of base space X.

But some statistical experiments more depend on the process of human mentality. The man who is an object of experiment and always uses a certain subset A todescribe fuzzy object, with $\delta(A)$ ($0 \leq \delta(A) \leq 1$) to represent the degree of self-confidence for A.

In the different degree of self-confidence he can use a set of different certain subset A to describe the same fuzzy object, and also can use fuzzy subset on X to describe it approximately. This method stated above can give the more information for us to describe fuzzy object objectively . We call this kind of statistical experiment the fuzzy set-valued statistics and call the method suggesting in [1] as $\mathbb{P}$-set-valued statistics. So that, we can enlarge the concept of random set and projected function in paper [1]. And we suggest the concept of fuzzy random set and projected function of the fuzzy random set. These are the theoretical base of the fuzzy set-valued statistics.

In this paper we have also proved the relation between random set and probabilistic set suggested by K. Hirota. In reality,the random set suggested in [1] may be considered as a particalar probabilistic set.

2. THE FUZZY RANDOM AND THE PROJECTED FUNCTION

In this section, we begin by introducting some concepts of the random set and the projected function and then extend these to the fuzzy random set and the projected function of the fuzzy random set, so that the random set and its projected function is a special case of the fuzzy random set and its projected function.

Let $\mathbb{P}(X)$ be the power set of a base space X and $\check{\mathbb{B}}$ σ-algebra containg the class X on $\mathbb{P}(X)$, where

$$\begin{aligned} \dot{X} &\triangleq \{\dot{x} \mid x \in X\}, \\ \dot{x} &\triangleq \{B \mid B \in \mathbb{P}(X),\ B \ni x\}. \end{aligned} \qquad (1)$$

Let $(\Omega, \mathscr{A}, P)$ be a given probability space. ($\mathbb{P}(X)$, $\check{\mathbb{B}}$) is a measurable space, and we call a mapping

$$\xi : \Omega \longrightarrow \mathbb{P}(X) \qquad (2)$$

a random set of X (or $\mathbb{P}$-random set), if $\forall B \in \check{\mathbb{B}}$, has

$$\xi^{-1}(B) = \{\omega \mid \xi(\omega) \in B\} \in \mathcal{A} .$$

Suppose that ξ is a random set of X from a given probability space $(\Omega, \mathcal{A}, P)$, for all $x \in X$,

$$\mu_\xi(x) \triangleq P(\omega \mid \xi(\omega) \ni x). \qquad (3)$$

we call that $\mu_\xi(x)$ is a projected function of ξ.

The collection of all fuzzy subset of X is denoted by $\mathbb{F}(X)$. Let $B(x)$ be the membership function of the fuzzy subset B, x_λ the fuzzy point [6] on X.

Definition 2.1 For fixed $x \in X$, we call

$$I_x \triangleq \{x_\lambda \mid \lambda \in [0,1]\} \triangleq \{x\} \times [0,1] \qquad (4)$$

the pointwise set of x.

We denote

$$\{\dot{x}_\lambda\}^c \triangleq \{B \mid B \in \mathbb{F}(X),\ B(x) \le \lambda\} \qquad (5)$$

$$\mathring{X} \triangleq \{\{\dot{x}_\lambda\}^c \mid x_\lambda \text{ is the fuzzy point on } X\} \qquad (6)$$

Definition 2.2 Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\mathbb{F}(X), \hat{\mathbb{B}})$ a measurable space, where $\hat{\mathbb{B}}$ is a smallest σ-algebra generated by the class X on $\mathbb{F}(X)$. We call a mapping

$$\xi : \Omega \longrightarrow \mathbb{F}(X)$$

a fuzzy random set of X, if $\forall \mathcal{G} \in \hat{\mathbb{B}}$, $\{\omega \mid \xi(\omega) \in \mathcal{G}\} \in \mathcal{A}$.

The collection of all fuzzy random sets of X is denoted by $R(\Omega, \mathcal{A}; \mathbb{F}(X), \hat{\mathbb{B}})$.

Definition 2.3 Suppose that β_x is a Borel set on the pointwise set I_x, ξ is a measurable mapping of $(\Omega, \mathcal{A})$ to $(\mathbb{F}(X), \hat{\mathbb{B}})$ and θ is also a measurable mapping of $(\mathbb{F}(X), \hat{\mathbb{B}})$ to (I_x, β_x). We call the composition mapping

$$\xi|x \triangleq \theta \circ \xi : \Omega \longrightarrow I_x$$

a restricted random variable of ξ at $x \in X$. If $(\Omega, \mathcal{A}, P)$ is a probability space, the real function on I_x

$$F_{\xi|x}(\alpha) \triangleq P(\omega \mid [\xi(\omega)](x) \le \alpha) = P(\omega \mid \xi(\omega) \in \{\dot{x}_\alpha\}^c) \quad (8)$$

is called a restricted distribution function of ξ at $x \in X$, and

$$\mu_\xi(x) = E(\xi|x) = \int_{[0,1]} \alpha \, dF_{\xi|x}(\alpha) \quad (9)$$

a projected function of ξ.

Definition 2.4 For fixed $(x,y) \in X \times Y$, we call that

$$I_{(x,y)} \triangleq \{(x,y)_\alpha \mid \alpha \in [0,1]\} \quad (10)$$

is a two-dimensional pointwise set of (x,y), where $(x,y)_\alpha$ is the fuzzy points on $X \times Y$.

Definition 2.5 For given two fuzzy random set $\xi : \Omega \longrightarrow \mathbb{F}(X)$ and $\eta : \Omega \longrightarrow \mathbb{F}(Y)$, the real function on $I_{(x,y)} \triangleq [0,1]$

$$F_{(\xi|x,\, \eta|y)}(\alpha) = P(\omega \mid [\xi(\omega)](x) \le \alpha,\ [\eta(\omega)](y) \le \alpha) \quad (11)$$

is called a joint distribution function, where $\xi|x$ and $\eta|y$ are respectively the restricted random variables of ξ and η.

Definition 2.6 The joint projected function of the fuzzy random sets ξ and η is denoted by

$$\mu_{(\xi,\eta)}(x,y) = \int_{[0,1]} \alpha \, dF_{(\xi|x,\, \eta|y)}(\alpha) \quad (12)$$

It is easy to show that the form (9) and (12) are respectively a extension of some correspondingly concepts.

3. THE FUZZY RANDOM SETS AND THE PROBABILISTIC SETS

K. Hirota (1975) suggested the concept of probabilistic set. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\Omega_c, \mathcal{B}_c) = ([0,1]$, Borel sets) is called a characteristic space. A probabilistic set A on X is defined by a mapping

$$A : X \times \Omega \longrightarrow \Omega_c \quad (13)$$
$$(x,\omega) \longmapsto A(x,\omega).$$

where $A(x,\cdot)$ is the $(\mathcal{A}, \mathcal{B}_c)$-measurable function for each fixed

$x \in X$.

The collection of all probabilistic sets on X is denoted by $M(\Omega, \mathscr{A}; \Omega_c, \beta_c)$.

Definition 3.1 Let $A \in M(\Omega, \mathscr{A}; \Omega_c, \beta_c)$, for fixed $\omega \in \Omega$, we define

$$B_\omega(x) \triangleq B(x) \triangleq A(\cdot, \omega). \tag{14}$$

The fuzzy set B which is taken $B(x)$ as the membership function is called a fuzzy set generated by the probabilistic set A.

Consider the mapping

$$\theta: \mathbb{F}(X) \longrightarrow X \times [0,1] \qquad B \mapsto \theta(B) = B(x) \tag{15}$$

Proposition 3.1 For a given probability space $(\Omega, \mathscr{A}, P)$, the mapping θ can induce a mapping of $R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$ to $M(\Omega, \mathscr{A}; \Omega_c, \beta_c)$.

$$\theta: R \rightarrow M.$$

Proof: $\forall \xi \in R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, we have $\theta(\xi) = [\xi(\cdot)](x)$. It is easy to see that $\theta(\xi)$ is $(\mathscr{A}, \beta_c)$-measurable function for any x $(\in X)$, and we denote $[\xi(\cdot)](x) = A(\cdot, \omega) \in M(\Omega, \mathscr{A}; \Omega_c, \beta_c)$.

Proposition 3.2 The mapping θ is a surjection of $R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$ to $M(\Omega, \mathscr{A}; \Omega_c, \beta_c)$.

Proof: $\forall A \in M(\Omega, \mathscr{A}; \Omega_c, \beta_c)$, $\exists \xi \in R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, so that $\theta(\xi) = [\xi(\cdot)](x)$ where $\xi(\omega)$ is a fuzzy set generated by the probabilistic set A for a given $\omega \in \Omega$, i.e., $[\xi(\cdot)](x) = A(\cdot, \omega)$. This completes our proof.

By Proposition 3.2, the mapping θ can obtain a equivalent relation $\sim$ on $R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, for any $\xi, \eta \in R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, $\xi \sim \eta \Longleftrightarrow \theta(\xi) = \theta(\eta)$. $\bar{R} = R/\sim$ denote the quotient set of $R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$ relative to the relation $\sim$.

Let $\sim'$ be an equivalent relation on $R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$ defined by the fuzzy random sets which have the same projected function, i.e.,

$\xi \sim' \eta \Longleftrightarrow \mu_\xi = \mu_\eta$. We denote $\overline{R}' = R/\sim'$.

Because of any $x \in X$, if $\xi \sim \eta$, then $\theta(\xi) = \xi|x = \eta|x = \theta(\eta)$, and $\mu_\xi = E(\xi|x) = E(\eta|x) = \mu_\eta$ for any $\xi, \eta \in R(\Omega, \mathcal{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, thus $\xi \sim' \eta$. This shows that

$$\xi \sim \eta \Rightarrow \xi \sim' \eta. \tag{16}$$

From (16) we can assert that the partition $\overline{R}$ is finer than $\overline{R}'$. In addition, all operations and characters relative to the probabilistic set may be in accord with corresponding operations and characters on the quotient set $\overline{R}$.

4. FUZZY SET-VALUED STATISTICS

In paper [1] , the concept of the set-valued statistics is given. Which discussed theoretically only the statisyical experiments in which all samples are the general subsets. We may extend the set-valued statistics to the fuzzy set-valued statistical experiments in which the samples may be fuzzy subsets.

For a given $\xi \in R(\Omega, \mathcal{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, we make n independent observation to ξ, obtained a set of samples:

$$u_1, u_2, \ldots, u_n \qquad (u_i \in \mathbb{F}(X),\ i = 1, 2, \ldots, n)$$

Thus, the samples $u_1, u_2, \ldots, u_n$ are n fuzzy subsets prossessed the same distribution as ξ. For any $x \in X$, we call that

$$\overline{u}(x) \triangleq \frac{1}{n}\sum_{i=1}^{n} u_i(x) \tag{17}$$

is the fuzzy covering frequency of ξ at x. When u_i $(i=1,2,\ldots,n)$ are all general subsets, then

$$\overline{u}(x) = \frac{1}{n}\sum_{i=1}^{n} \chi_{u_i}(x). \tag{18}$$

Where $\chi_{u_i}(x)$ is a characteristic function of subset u_i. Using (17), we may estimate the projected value of ξ at x.

Theorem 4.1 Suppose that $\xi_1, \xi_2, \ldots$ are same distributional fuzzy random sets, and $\mu_{\xi_i}(x) = \mu(x)$, then we have

$$\frac{1}{n}\sum_{i=1}^{n}[\xi_i(\omega)](x) \longrightarrow \mu(x). \qquad \text{a .e. p } (\, n \to \infty \,)$$

By (9) and Kolmogorov Law of large numbers, Theorem 4.1 is obvious.

Let m be a positive valued measure on the measurable space $(X, \mathbb{B})$ and the projected function μ_ξ of the fuzzy random set ξ on X be integrable for $\mathbb{B}$. We denote

$$\overline{m}(\mu_\xi) = \int \mu_\xi(x) m(dx) \tag{19}$$

Obviously, if m is the Lebesgue measure on X, then geometric significance of $\overline{m}(\mu_\xi)$ is an area enclosed by the projected curve μ_ξ and axis of abscissas OX. We call that $\overline{m}(\mu_\xi)$ the projected cardinal.

Theorem 4.2 Let $(X, \mathbb{B})$ be a measurable space, m be a positive valued measure. For $\xi \in R(\Omega, \mathscr{A}; \mathbb{F}(X), \hat{\mathbb{B}})$, if $\mu(x,\omega) = [\xi(\omega)](x)$ is $\mathbb{B} \times \mathscr{A}$ measurable, then

$$\overline{m}(\mu_\xi) = E(\ \overline{m}(\ \xi(\omega))).$$

Proof: Using Fubini theorem, we have

$$\begin{aligned}
\overline{m}(\mu_\xi) &= \int_X \mu_\xi(x) m(dx) = \int_X \Big[\int_{[0,1]} \alpha \, dF_{\xi|x}(\alpha)\Big] m(dx) \\
&= \int_X \Big[\int_\Omega [\xi(\omega)](x) P(d\omega)\Big] m(dx) \\
&= \int_X \Big[\int_\Omega \mu(x,\omega) P(d\omega)\Big] m(dx) \\
&= \int_\Omega \Big[\int_X \mu(x,\omega) m(dx)\Big] P(d\omega) \\
&= \int \overline{m}(\xi) P(d\omega) = E(\ \overline{m}(\xi))
\end{aligned}$$

Definition 4.1 Let $\xi : \Omega \to \mathbb{F}(X)$, $\forall x \in X$

$$D_\xi(x) = 4D(\xi \mid x) = 4\int_{[0,1]} (\alpha - E(\xi \mid x))^2 dF_{\xi|x}(\alpha) \tag{20}$$

call the degree of stability of the fuzzy random set ξ.

Property: $0 \le D_\xi(x) \le 1$.

Proof: $0 \le D_\xi(x)$ is obvious. We will show only $D_\xi(x) \le 1$. i.e.,

$D(\xi|x) \le \frac{1}{4}$.

Because $D(\xi|x) = E[(\xi|x)^2] - E^2(\xi|x) \le E(\xi|x) - E^2(\xi|x)$, we take $E(\xi|x) = y$, and consider the maximum of the function $w = y - y^2$ on the interval $[0,1]$. Let

$$\frac{dw}{dy} = 1-2y = 0,$$

we obtain that $y = \frac{1}{2}$. Hence maximum $w = \frac{1}{4}$. Thus

$$D(\xi|x) \le \frac{1}{4}.$$

It shows that $\frac{1}{4}$ is a upper bound. But we consider a particular distribution of $\xi|x$ as follows;

$\xi|x$	0	1
P	$\frac{1}{2}$	$\frac{1}{2}$

We compute $D(\xi|x) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Hence $\frac{1}{4}$ is a minimum upper bound.

Definition 4.2 Let m be a positive valued measure on the measurable space ($X, \mathbb{B}$), and the stable function $D_\xi(x)$ of the fuzzy random set ξ on X is integrable for $\mathbb{B}$, We call that

$$\sigma_\xi \triangleq \int D_\xi(x)m(dx) \,/\, \overline{m}(\mu_\xi) \tag{21}$$

the mean degree of stability.

In the fuzzy set-valued statistics, we can compute $D_\xi(x)$ and σ_ξ with the methods of parameter estimation of the general theory of statistics. When $D_\xi(x) = 0$, we say that x ($\in X$) is the most stable for ξ ; and $D_\xi(x) = 1$, the most instable for ξ . When $\sigma_\xi = 0$, we say that ξ is the most stable, and $\sigma_\xi = 1$, the most instable. The mean degree of stability σ_ξ can express the understanding level of the experimenter for the intension of descriptive object. And the degree of stability $D_\xi(x)$ expresses the understanding level for each part $x \in X$.

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