FIXED POINTS IN FUZZY METRIC SPACES

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Abstract. In this note the well-known fixed point theorems of Banach and Edelstein are extended to fuzzy metric spaces in the sense of Kramosil and Michalek.

The Banach fixed point theorem states that each selfmapping T of a complete metric space (X, d) such that $d(Tx, Ty) < k \cdot d(x, y)$ $(x \neq y,\ 0 < k < 1)$ has a unique fixed point. The assumption $k < 1$ is nonsuperfluous. With $k = 1$ the mapping of that sort need not have a fixed point. However, if X is compact, then T has a unique fixed point (Edelstein [3]).

In this note we extend fixed point theorems of Banach and Edelstein to contractive mappings of a complete and compact fuzzy metric space, respectively. We shall deal with fuzzy metric spaces introduced by Kramosil and Michalek [6]. Note there are at least five different concepts of a fuzzy metric space (cf. [1], [2], [4]-[6]).

We begin with some definitions.

1. DEFINITION ([8]). A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a (continuous) t-norm if $([0, 1], *)$ is an abelian (topological) monoid with the unit 1 such that $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$ $(a, b, c, d \in [0, 1])$.

2. DEFINITION ([6]). The 3-tuple $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(2.1) $M(x, y, 0) = 0$,

(2.2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,

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(2.3) $M(x, y, t) = M(y, x, t)$,

(2.4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,

(2.5) $M(x, y, \cdot): [0, \infty) \to [0, 1]$ is left-continuous,

for all $x, y, z \in X$ and $t, s > 0$.

3. DEFINITION. A sequence $\{x_n\}$ in X is Cauchy if $\lim_n M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p > 0$. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_n M(x_n, x, t) = 1$ for each $t > 0$. Notation: $\lim_n x_n = x$. (Since $*$ is continuous, it follows from (2.4) that the limit is uniquely determined).

A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.

4. LEMMA. $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Proof. Suppose $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$. Then $M(x, y, t) * M(y, y, s-t) \leq M(x, y, s) < M(x, y, t)$. By (2.2), $M(y, y, s-t) = 1$, thus $M(x, y, t) < M(x, y, t)$, a contradiction.

Note. Kramosil and Michalek [6] assumed actually $*$ to be measurable only, and, consequently, they assumed $M(x, y, \cdot)$ to be non-decreasing. Note also that the condition (5.1) below is included in their definition of a fuzzy metric space.

5. THEOREM (fuzzy Banach contraction theorem). Let $(X, M, *)$ be a complete fuzzy metric space such that

(5.1) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$.

Let $T: X \to X$ be a mapping satisfying

(5.2) $M(Tx, Ty, kt) \geq M(x, y, t)$

for all $x, y \in X$ where $0 < k < 1$. Then T has a unique fixed point.

Proof. Let $x \in X$ and let $x_n = T^n x$ ($n \in \mathbb{N}$). By a simple induction we get

(5.3) $$M(x_n, x_{n+1}, kt) \geq M(x, x_1, \frac{t}{k^{n-1}})$$

for all n and $t > 0$. Thus for any positive integer p we have

$$M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, \frac{t}{p}) * \overset{(p)}{\cdots} * M(x_{n+p-1}, x_{n+p}, \frac{t}{p}) \geq$$

$$\geq M(x, x_1, \frac{t}{pk^n}) * \overset{(p)}{\cdots} * M(x, x_1, \frac{t}{pk^n})$$

by (5.3). According to (5.1) we now have

$$\lim_n M(x_{n+p}, x_n, t) \geq 1 * \overset{(p)}{\cdots} * 1 = 1,$$

i.e., $\{x_n\}$ is Cauchy, hence convergent. Call the limit y. Thus we have

$$M(Ty, y, t) \geq M(Ty, Tx_n, \frac{t}{2}) * M(x_{n+1}, y, \frac{t}{2})$$

$$\geq M(y, x_n, \frac{t}{2k}) * M(x_{n+1}, y, \frac{t}{2}) \to 1 * 1 = 1$$

by (2.4). By (2.2) we get $Ty = y$, a fixed point. To show uniqueness, assume $Tz = z$ for some $z \in X$. Then

$$1 \geq M(z, y, t) = M(Tz, Ty, t) \geq M(z, y, \frac{t}{k}) = M(Tz, Ty, \frac{t}{k})$$

$$\geq M(z, y, \frac{t}{k^2}) \geq \dots \geq M(z, y, \frac{t}{k^n}) \to 1$$

as $n \to \infty$. By (2.2), $z = y$.

6. <u>LEMMA</u>. If $\lim_n x_n = x$ and $\lim_n y_n = y$, then

$$M(x, y, t-\varepsilon) \leq \lim_n M(x_n, y_n, t) \leq M(x, y, t+\varepsilon)$$

for all $t > 0$ and $0 < \varepsilon < \frac{t}{2}$.

P r o o f. By (2.4), $M(x_n, y_n, t) \geq M(x_n, x, \frac{\varepsilon}{2}) * M(x, y, t-\varepsilon) * M(y, y_n, \frac{\varepsilon}{2})$. Thus, $\lim_n M(x_n, y_n, t) \geq 1 * M(x, y, t-\varepsilon) * 1 = M(x, y, t-\varepsilon)$. On the other hand, $M(x, y, t+\varepsilon) \geq M(x, y_n, \frac{\varepsilon}{2}) * M(x_n, y_n, t) * M(y_n, y, \frac{\varepsilon}{2})$, hence $M(x, y, t+\varepsilon) \geq \lim_n M(x_n, y_n, t)$. So, the assertion follows.

7. <u>COROLLARY</u>. Let $\lim_n x_n = x$ and $\lim_n y_n = y$. Then:

(7.1) $\lim_n M(x_n, y_n, t) \geq M(x, y, t)$ for all $t > 0$;

(7.2) If $M(x, y, \cdot)$ is continuous, then $\lim_n M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.

8. <u>THEOREM</u> (fuzzy Edelstein contraction theorem). Let $(X, M, *)$ be a compact fuzzy metric space with $M(x, y, \cdot)$ continuous for all $x, y \in X$. Let $T: X \to X$ be a mapping satisfying

(8.1) $M(Tx, Ty, t) > M(x, y, t)$

for all $x \neq y$ and $t > 0$. Then T has a unique fixed point.

P r o o f. Let $x \in X$ and $x_n = T^n x$ $(n \in \mathbb{N})$. Assume $x_n \neq x_{n+1}$ for each n (if not, $Tx_n = x_n$). Now, assume $x_n \neq x_m$ $(n \neq m)$. For otherwise we get $M(x_n, x_{n+1}, t) = M(x_m, x_{m+1}, t) > M(x_{m-1}, x_m, t) > \ldots > M(x_n, x_{n+1}, t)$ where $m > n$, a contradiction. Since X is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$. Let $y = \lim_i x_{n_i}$. We also assume that $y, Ty \notin \{x_n : n \in \mathbb{N}\}$ (if not, choose a subsequence with such a property). According to the above assumptions we may now write

$$M(Tx_{n_i}, Ty, t) > M(x_{n_i}, y, t)$$

for all $i \in \mathbb{N}$ and $t > 0$. Since $M(x, y, \cdot)$ is continuous for all x, y in X, by (7.2) we obtain

$$\lim_i M(Tx_{n_i}, Ty, t) \geqslant \lim_i M(x_{n_i}, y, t) = 1$$

for each $t > 0$, hence

(8.2) $$\lim_i Tx_{n_i} = Ty .$$

Similarly, we obtain

(8.3) $$\lim_i T^2 x_{n_i} = T^2 y$$

(recall that $Ty \neq Tx_{n_i}$ for all i). Now, observe that $M(x_{n_1}, Tx_{n_1}, t) < M(Tx_{n_1}, T^2x_{n_1}, t) < \ldots < M(x_{n_i}, Tx_{n_i}, t) < M(Tx_{n_i}, T^2x_{n_i}, t) < \ldots < M(x_{n_{i+1}}, Tx_{n_{i+1}}, t) < M(Tx_{n_{i+1}}, T^2x_{n_{i+1}}, t) < \ldots < 1$ for all $t > 0$. Thus $\{M(x_{n_i}, Tx_{n_i}, t)\}$ and $\{M(Tx_{n_i}, T^2x_{n_i}, t)\}$ $(t > 0)$ are convergent to a common limit (cf. [7]). So, by (8.2), (8.3) and (7.2) we get

$$M(y, Ty, t) = M(\lim x_{n_i}, T(\lim x_{n_i}), t)$$

$$= \lim M(x_{n_i}, Tx_{n_i}, t)$$

$$= \lim M(Tx_{n_i}, T^2x_{n_i}, t)$$
$$= M(\lim Tx_{n_i}, \lim T^2x_{n_i}, t)$$
$$= M(Ty, T^2y, t)$$

for all $t > 0$. Suppose $y \neq Ty$. Then, by (8.1), $M(y, Ty, t) < M(Ty, T^2y, t)$ $(t > 0)$, a contradisction. Hence $y = Ty$, a fixed point. Uniqueness follows at once from (8.1).

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