PRIME L-FUZZY AND PRIMARY L-FUZZY IDEALS

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Dedicated to my wife

In this paper we introduced the concepts of the primary L-fuzzy ideal and the prime L-fuzzy ideal factor belouging to the primary L-fuzzy ideal, and proved some fundamental propositions. In particular, it gave a structural theorem of a prime L-fuzzy ideal factor belonging to a primary L-fuzzy ideal.

Keywords: Base set, Prime L-fuzzy ideal, Primary L-fuzzy ideal.

1. Introduction

The concept of fuzzy groups was introduced by Rosenfeld in [1]. Several orther authors continued the investigation of such concepts, as [2,3,4].

In [4], we defined also maximum fuzzy ideals and prime fuzzy ideals. In this paper we continue the program begun in [4], and we extend to the case of L-fuzzy sets, where L is a completely distributive lattice.

In Section 3, we introduced the concepts of the primary L-fuzzy ideal and a prime L-fuzzy ideal factor belonging to a primary L-fuzzy ideal, and prove some fundamental propositions. In particular, it gave a structural theorem of a prime L-fuzzy ideal factor belonging to a prime L-fuzzy ideal.

2. L-fuzzy Ideals and Prime L-fuzzy Ideals

We first recall some basic concepts occurring in the papers [4], and we extend to case of L-fuzzy sets.

Throughout this paper $L=(L, \leq, \wedge, \vee)$ will be a completive distributive lattice, which has the least and greatest elements, say 0 and 1, respectively. Let X be a non-empty (usual) set. An L-fuzzy set in X is a map $A: X \longrightarrow L$, and $F_L(X)$ will denote the set of all L-fuzzy sets in X.

It is easily seen that $F_L(X) = (F_L(X), \leq, \wedge, \vee)$ is a completely distributive lattice, which has the least and greatest elements, say $\mathbf{0}$ and $\mathbf{1}$, respectively in a natural manner, where $\mathbf{0}(x)=0$, $\mathbf{1}(x)=1$ for any $x \in X$.

Definition 2.1 Let $X=(X, +, \cdot)$ be a ring, and $A \in F_L(X)$, $A \neq \mathbf{0}$. A is called an L-fuzzy subring of X, if A is an L-fuzzy subgroup under the binary operation '$\oplus$' in $F_L(X)$ induced by '+' in X, and A is an L-fuzzy subgroupoid under '$\odot$' in $F_L(X)$ induced by '$\cdot$' in X (cf.[3]).

Definition 2.2 Let X be a ring, and $A \in F_L(X)$, $A \neq \mathbf{0}$. A will be called an L-fuzzy ideal, iff

(1) A is an L-fuzzy subgroup of X under '$\oplus$';

(2) $\mathbf{1} \odot A \leq A$, $A \odot \mathbf{1} \leq A$.

(cf. 3).

Now let $I_L(X) \subseteq F_L(X)$ be the set of all L-fuzzy ideals in X.

Proposition 2.1 Let X be a ring, $A \in F_L(X)$, $A \neq 0$. Then $A \in I_L(X)$ iff the level subsets A_t; for any $t \in L$, $t \leq A(0)$, is an ideal of X.

(The proof is omitted).

Definition 2.3 Let $A \in I_L(X)$, The ideals A_t, for $t \in L$ and $t \leq A(0)$, are called level ideals of A.

Theorem 2.1 Let H, N be ideals of a ring X and $H \subseteq N$. Then there exists $A \in I_L(X)$ such that

$$A_t = H, \qquad A_k = N.$$

where t, $k \in L$ and $t \geqslant k$.

Proof. Let $A \in F_L(X)$ defined by

$$A(x) = \begin{cases} k, & \text{if } x \in N \setminus H. \\ t, & \text{if } x \in H, \\ 0, & \text{if } x \in X \setminus N, \ t, k \in L \text{ and } t \geqslant k. \end{cases}$$

We have, for $l \in L$,

$$A_l = \begin{cases} A_0 = X, & \text{if } l = 0. \\ A_k = N, & \text{if } k \geqslant l > 0. \\ A_t = H, & \text{if } t \geqslant l > k. \\ \emptyset, & \text{if } 1 \geqslant l > t. \end{cases}$$

It follows that every nonempty level suhset of A is an ideal of X. Hence A is an L-fuzzy ideal of X from Proposition 2.1. Clearly, A satisfies the conditions of the theorem.

Let H=N. We obtain as follows:

Corollary 2.1 Let H be any ideal of a ring X. Then there exists $A \in I_L(X)$ such that level ideal $A_t=H$, where $t \in L$.

Definition 2.4 Let A be an L-fuzzy subring of the ring X, the set

$$X_A = \{x \in X \mid A(x)=A(0)\}$$

is called a base set of A.

We have, from Proposition 2.1.

Proposition 2.2 If $A \in I_L(X)$, then the base set X_A is an ideal of X.

We will follow a convention that whenever ring X is tacitly assumed to be commutative.

Definition 2.5 Let $A \in I_L(X)$. A is called a prime L-fuzzy ideal, if for $a,b \in X$, $A(ab)=A(0)$ implies $A(a)=A(0)$ or $A(b)=A(0)$, where $0 \in X$ is the null element of X.

Theorem 2.2 Let $A \in I_L(X)$. Then A is a prime L-fuzzy ideal of X iff,

X_A is a prime ideal of X.

(cf. [4]).

3. Primary L-fuzzy Ideals

Definition 3.1 Let $A \in I_L(X)$. A is called a primary L-fuzzy ideal of X, if for $a,b \in X$, $A(ab)=A(0)$ and

$A(a) \neq A(0)$ implies $A(b^n)=A(0)$ for some $n \in \mathbb{N}$, where $\mathbb{N}$ is the natural number set.

We have, from Definition 2.5 and Definition 3.1, every prime L-fuzzy ideal is all a primary L-fuzzy ideal.

Definition 3.2 Let $A, B \in I_L(X)$. B is called a factor of A, if $X_A \subseteq X_B$.

Definition 3.3 A prime L-fuzzy ideal P is called a prime L-fuzzy ideal factor belonging to primary L-fuzzy ideal Q, if

(1) P is a factor of Q.

(2) For $b \in X$, $P(b)=P(0)$ iff $Q(b^n)=Q(0)$ for some $n \in \mathbb{N}$.

Theorem 3.1 Let P be a L-fuzzy subring of X and Q be a primary L-fuzzy ideal of X. If $P(b)=P(0)$, for $b \in X$, iff $Q(b^n)=Q(0)$ for some $n \in \mathbb{N}$. Then P is a prime L-fuzzy ideal factor belonging to Q.

Proof. (1) $P \in I_L(X)$. Since $Q(b^n)=Q(0)$, i.e. $b^n \in X_Q$ and X_Q is an ideal in X from Proposition 2.2, $(xb)^n \in X_Q$ for any $x \in X$, i.e. $Q((xb)^n)=Q(0)$. Therefore $xb \in X_P$. Let $b, c \in X$ and $Q(b^m)=Q(0)$ and $Q(c^l)=Q(0)$, i.e. b^m, $c^l \in X$. Since every term of the expansion of the binomial $(b-c)^{m+l-1}$ contains either b^m or c^l, $(b-c)^{m+l-1} \in X_Q$. Therefore $(b-c) \in X$. We have, from Proposition 2.1, P is an L-fuzzy ideal.

(2) We prove that P is a prime L-fuzzy ideal. Let $a,b \in X$ and $P(ab)=P(0)$ and $P(a)\neq P(0)$. It follows that $Q((ab)^n)=Q(0)$ and $Q(a^n)\neq Q(0)$ for some n. Since Q is a primary L-fuzzy ideal, $Q(b^{nm})=Q(0)$ for some m. Therefore $P(b)=P(0)$.

(3) P is a factor of Q, i.e. $X_Q \subseteq X_P$. Since elements of X_Q have the property which some power of elements of X_Q belong to X_Q.

The following Theorem will give a structural property of the prime L-fuzzy ideal factor P belonging to the primary L-fuzzy ideal Q.

Theorem 3.2 Let $P,Q \in I_L(X)$. Then every primary L-fuzzy ideal Q of X there exists a prime L-fuzzy ideal factor P, iff

(1) P is a factor of Q.

(2) If $P(b)=P(0)$ for $b \in X$, then $Q(b^n)=Q(0)$ for some $n \in \mathbb{N}$.

(3) If $Q(ab)=Q(0)$ and $Q(a)\neq Q(0)$ for $a,b \in X$, then $P(b)=P(0)$.

Proof. The necessity can be directly verified. Sufficiency. i) Q is a primary L-fuzzy ideal. Let $a,b \in X$ and let $Q(ab)=Q(0)$ and $Q(a)\neq Q(0)$. We have, from conditions (2) and (3), there exists some $n \in \mathbb{N}$ such that $Q(b^n)=Q(0)$. Therefore Q is a primary L-fuzzy ideal. ii) The existence proof of P. Let P be defined as follows:

$$P(b)=\begin{cases} P(0)=t, & \text{iff } b^n\in X_Q,\ t\in L. \\ 0, & \text{iff } b^n\notin X_Q,\ 0\in L. \end{cases}$$

We have, from Theorem 2.1, $P\in I_L(X)$.

iii) P is a prime L-fuzzy ideal factor. Let $b\in X$ and $P(b)=P(0)$. We have, from the condition (2), there exists some $n\in\mathbb{N}$ such that $Q(b^n)=Q(0)$. Now we verify that $Q(b^n)=Q(0)$ implies $P(b)=P(0)$. Let r is the least natural number such that $Q(b^n)=Q(0)$ for $b\in X$. If r=1, from the condition (1), the statement is true. Now let $r>1$, it follows that $Q(b^r)=Q(bb^{r-1})$ $=Q(0)$ and $Q(b^{r-1})\neq Q(0)$, We have, from (3), $P(b)=P(0)$. P is a prime L-fuzzy ideal factor belonging to Q from Theorem 3.1.

Remark. Theorem 3.2, in some particular conditions, reduces a primary L-fuzzy ideal Q which is verified, and reduces the prime L-fuzzy ideal factor P belonging to Q which is found, and point out the prime L-fuzzy ideal factor P belonging to Q to be uniquely determined by the conditions of Theorem 3.2.

Theorem 3.3 P is a prime L-fuzzy ideal factor belonging to the primary L-fuzzy ideal Q of X iff X_P is a prime ideal factor belonging to the primary ideal X_Q of X.

Proof. Let P be a prime L-fuzzy ideal factor belonging to the primary L-fuzzy ideal Q, then the conditions (1), (2) and (3) of Theorem 3.2 are satisfied, iff the conditions (1) X_P is a factor of X_Q, (2) If $b \in X_P$, then $b^n \in X_Q$ for some $n \in \mathbb{N}$, and (3) If $ab \in X_Q$ and $a \notin X_Q$, then $b \in X_P$ are satisfied. Therefore X_P is a prime ideal factor belonging to the primary ideal X_Q.

Theorem 3.4 If P is a prime L-fuzzy ideal factor belonging to both primary L-fuzzy ideal Q_1 and Q_2 of X. Then P is yet a prime L-fuzzy ideal factor belonging to the primary L-fuzzy ideal $Q=Q_1 \wedge Q_2$ of X.

Proof. (1) Clearly, P is a factor of Q.

(2) Let $a,b \in X$ and let $Q(ab)=Q(0)$ and $Q(a) \neq Q(0)$. We have at least a primary L-fuzzy ideal, and suppose Q_1 such that $Q_1(ab)=Q_1(0)$ and $Q_1(a) \neq Q_1(0)$. Therefore $P(b)=P(0)$.

(3) Let $b \in X$ and $P(b)=P(0)$. Since Q_1 and Q_2 are primary L-fuzzy ideals of X, $Q_1(b^l)=Q_1(0)$ and $Q_2(b^m)=Q_2(0)$ for $l,m \in \mathbb{N}$. Suppose $n=\max(l,m)$, then $Q_1(b^n)=Q_1(0)$ and $Q_2(b^n)=Q_2(0)$. Therefore $Q(b^n)=Q(0)$. We have, from Theorem 3.2, P is yet a prime L-fuzzy ideal factor belonging to the primary L-fuzzy ideal $Q=Q_1 \wedge Q_2$ of X.

Corollary 3.1 If P is a prime L-fuzzy ideal factor belonging to all of primary L-fuzzy ideals Q_1, Q_2, ..., Q_m of X. Then P is yet a prime L-fuzzy ideal factor belonging to the primary L-fuzzy ideal $Q=\bigwedge_{i=1}^{m} Q_i$ of X.

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