RELATIONSHIP BETWEEN NORMAL HYPERGROUPS AND QUOTIENT GROUPS

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The relationship between the concepts of hypergroups proposed in Ref.(1) and quotient groups is discussed here, a main consequence is that: every normal hypergroup must be a generalized quotient group.

KEYWORDS: Hypergroup, Generalized quotient group, Quotient group, Group.

1. INTRODUCTION

The importance of set value mappings has been highlighted with researches in the theoretical basis of Fuzzy mathematics. All kinds of mathematical structures need to be promoted from their universes to their power sets. In Ref.(1) the concept of hrpergroup was proposed.

What is called a hypergroup is a group formed by that the operation in the group is induced into its power set. Let $(G,\cdot)$ be a group. By " $\cdot$ " we can naturally induce an operation in $2^G-\{\emptyset\}$ (denoted by " $\cdot$ ", too): For any $A, B \in 2^G-\{\emptyset\}$,

$$A\cdot B=\{a\cdot b \mid a\in A, b\in B\} \qquad (1)$$

So, the definition of hypergroups (see Ref.(1)) is that: $\mathcal{G} \subset 2^{G}-\{\emptyset\}$ is called a hypergroup if it is group with respect to the operation " · " of the formula (1).

Such definition of hypergroup is quite natural.

A example of hypergroup is quotient group fristly. The quotient group of G with respect to a normal subgroup N, G/N, is just a hypergroup on G. But there are hypergroups which aren't quotient groups. It is worth to clear what relationship is there between hypergroups and quotient groups.

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Now we extend the structure of quotient group a little. For any subgroup of G_o and a subset E, of G, with

1) $E^2=E$ (2)

2) $(\forall a\in G_o)(aE=Ea)$ (3)

It is easy to prove that $\mathcal{G} \triangleq \{aE \mid a\in G_o\}$ is a hypergroup on G, which its identity element is just E and the inverse element of aE is $a^{-1}E$. Here the sign of operation " · " is always omitted and $\{a\}N$ is simply denoted by aN.

According as Ref.(1) and extended a little we give

DEFINITION 1 A subsemigroup E of G is called normal subsemigroup of G, if

1) $e\in E$ (e is the identity element of G) (4)

2) For any subgroup G_o of G, it satisfies the formula (3).

Now wirte

$$G_o \mid E \triangleq \{aE \mid a\in G_o\} \quad (5)$$

it is called generalized group which G_o is with respect to E.

Since the subsemigroup satisfied the formula (4) must satisfy the formula (2), from above a consequence we have that generalized quotient groups $G_o|E$ must be hypergroups on G satisfied the formula (4).

DEFINITION 2 The hypergroups satisfied the formula (4) are called normal hypergroups.

A generalized quotient group is a normal hypergroup. This paper will prove that a normal hypergroup must also be a generalized group.

Let $\mathcal{G}$ be a hypergroup, $\forall A \in \mathcal{G}$, write

$$\overline{A} \triangleq \{a \in A \mid a^{-1} \in A^{-1}\} \qquad (6)$$

it is called Kernel of A.

LEMMA 1 If $\mathcal{G}$ is normal, then $\overline{A} \neq \emptyset$ for any $A \in \mathcal{G}$.

PROOF. From $AA^{-1}=E$ and $e \in E$ we know there are $a \in A$ and $b \in A^{-1}$ such that $ab=e$. So $b=a^{-1} \in A^{-1}$. Thus $a \in \overline{A}$. This means $\overline{A} \neq \emptyset$. Q.E.D.

LEMMA 2 If $\mathcal{G}$ is normal, then that $a \in \overline{A}$ implies

$$A=aE=Ea \qquad (7)$$

PROOF. Firstly it is clear that $aE \subset AE=A$. Secondly, for any $b \in A$, $a \in \overline{A}$, we have $b=eb=(aa^{-1})b=a(a^{-1})b$ $=a(a^{-1}b) \subset a(A^{-1}A)=aE$. So $A \subset aE$. Therefore $A=aE$. The proof which $A=Ea$ is similar. Q.E.D.

LEMMA 3 If $\mathcal{G}$ is a hypergroup on G, then

$$G_o \triangleq \cup\{\overline{A} \mid A \in \mathcal{G}\} \qquad (8)$$

is a subgroup of G.

PROOF. For any $a, b \in G_o$, there are $A, B \in \mathcal{G}$, such that $a \in \overline{A}$, $b \in \overline{B}$, i.e., $a \in A$, $a^{-1} \in A^{-1}$, $b \in B$, $b^{-1} \in B^{-1}$. Thus $ab^{-1} \in AB^{-1} \triangleq C \in \mathcal{G}$, and $(ab^{-1})^{-1}=ba^{-1} \in BA^{-1}=C^{-1}$. So $ab^{-1} \in \overline{C} \subset G_o$. Therefore G_o is a subgroup of G. Q.E.D.

From above the results we have

THEOREM 1 If $\mathcal{G}$ is a normal hypergroup on G, then it must be a generalized quotient group:

$$\mathcal{G} = G_o | E \qquad (9)$$

here E is the identity element of $\mathcal{G}$ which is a normal subsemigroup of G and G_o is just as the formula (8).

Normal hypergroups, i.e., generalized quotient groups, what relationships are there between them and quotient groups ?

When G_o=G and E is a normal subgroup of G, a generalized quotient group become a quotient group. Thus generalized quotient groups are more extended concept than quotient group.

Let $\mathcal{G}=G_o|E$ be a known normal hypergroup. Consider the mapping

$$f : G_o \longrightarrow \mathcal{G}$$

$$a \longmapsto aE \qquad (10)$$

LEMMA 4 $\quad Ker(f)=\overline{E} \qquad$ (11)

PROOF. Notice $Ker(f)=\{a\in G_o \mid aE=E\}$. For any $a\in Ker(f)$ we have aE=E. Since $e\in E$ and aE=E, $ae\in E$, so $a\in E$. From $e\in E$ and E=aE we know there is $b\in E$ such that e=ab, thus $b=a^{-1}$, therefore $a^{-1}\in E$. Now we know $a\in\overline{E}$, i.e., $Ker(f)\subset\overline{E}$.

For any $a\in\overline{E}$ we have aE=E from Lemma 2, so $a\in Ker(f)$, i.e., $\overline{E}\subset Ker(f)$. Q.E.D.

From this we have

LEMMA 5 If $\mathcal{G}$ is a normal hypergroup on G, then $\overline{E}$ is a normal subgroup of G_o .

Notice that the generalized quotient group $\mathcal{G}=G_o|E$ and the quotient group $G_o/\overline{E}\triangleq\overline{\mathcal{G}}$ are isomorphic.

The equivalence relation with respect to $G_o/\overline{E}$ is

$$a\sim b \text{ iff } a\overline{E}=b\overline{E} \text{ iff } aE=bE \qquad (12)$$

The equivalence class which a is in is denoted by (a). Clearly we have

$$(a)=a\overline{E}\subset aE \qquad (13)$$

$$\overline{aE}=aE \qquad (\forall a\in G_o) \qquad (14)$$

Thus we have

THEOREM 2 If $\mathcal{G} = G_o|E$ is a normal hypergroup on G, then it must contact a quotient group $\bar{\mathcal{G}} = G_o/\bar{E}$. They are isomorphic by the mapping

$$f : \mathcal{G} : \longrightarrow \bar{\mathcal{G}}$$

$$A \longmapsto \bar{A}$$

$$(aE \longmapsto a\bar{E}) \qquad (15)$$

$\mathcal{G}$ and $\bar{\mathcal{G}}$ decide the same classification. Each class must coincide with the coset of $\bar{E}$ and be contained the coset of $\bar{E}$ (see the formula (13)).

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