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A FUZZY MODIFICATION OF THE MAXIMAL ERGODIC THEOREM

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The paper is aimed to formulate and prove the fuzzy modification of the classical maximal ergodic theorem with respect to a fuzzy ordering of the range of an integrable function. The main idea of the proof of Theorem 1 given below is originally due to Garsia [1] . For detail studiing of the classical cases see e.g. [2] , [3],[4] . Other works of the related fuzzy topics are [5] , [6] , [7] .

Let $(X, \mathcal{S}, \lambda, \tau)$ be a dynamical system, i.e. X is a non-empty set, $\mathcal{S}$ is a σ-algebra on X, λ is a measure on $\mathcal{S}$ and $\tau : X \to X$ is a measure λ-preserving transformation. Let $\mathcal{R} : R^2 \to \langle 0, 1\rangle$ be a fuzzy relation of the linear ordering of the set of real numbers R defined as follows:

R 1) $\forall\ x, y \in R,\ x \neq y :\ \mathcal{R}(x,y) > 0 \Longleftrightarrow \mathcal{R}(y,x) = 0$

(antisymetry and connectivity)

R 2) $\forall\ x, y, z \in R:\ \mathcal{R}(x,z) \geq \sup_{y \in R} \mathcal{R}(x,y)\,\mathcal{R}(y,z)$ (transitivity)

Compatibility of the mentioned fuzzy-ordering by the natural ordering of the set R is given by the following conditions:

R 3) $\forall\ x \in R:\ \mathcal{R}(0,x) > 0 \Longleftrightarrow x > 0$

R 4) $\forall\ x, y, z \in R:\ \mathcal{R}(x,y) > 0 \Longrightarrow \mathcal{R}(x+z, y+z) > 0$

R 5) $\forall\ x, y, z \in R:\ \mathcal{R}(x,y) > 0,\ \mathcal{R}(y,z) > 0 \Longrightarrow \mathcal{R}(x,y) \leq \mathcal{R}(x,z)$

From the five conditions above it is easy to prove further properties of the relation $\mathcal{R}$:

R 6) $\forall\ x \in R:\ \mathcal{R}(x,x) = 0$

R 7) $\forall\ a \in R:\ \mathcal{R}(a,x)$ is a nondecreasing function of the variable x.

the fuzzy relation $\mathcal{R}$ will be called a membership function of the above defined fuzzy ordering. A simple nontrivial example of the above mentioned relation $\mathcal{R}$ is:

$$\mathcal{R}_\alpha(x,y) = \begin{cases} 1 - e^{\alpha(x-y)} & ; x < y \\ 0 & ; x \geq y \end{cases} \qquad \alpha > 0$$

Let $P = \{(x,y) \in R^2;\ x < y\}$. Evidently $\chi_P(x,y)$ is the membership function of the natural (non-fuzzy) ordering of the set R. It is easy to see that $\chi_P(x,y)$ is the limit case of the membership function $\mathcal{R}_\alpha(x,y)$ of the fuzzy ordering because of $\lim_{\alpha\to\infty} \mathcal{R}_\alpha(x,y) = \chi_P(x,y)$.

Let $f \in L_1(\lambda)$. Let us denote

$Tf: X \to R,\ x \mapsto f(\tau(x))$

$(S_0 f): X \to R,\ x \mapsto 0$

$(S_k f): X \to R,\ x \mapsto f(x) + f(\tau x) + \dots + f(\tau^{k-1} x)$ for $k = 1, 2, 3, \dots$

$(S_n^+ f): X \to R,\ x \mapsto \max_{0 \leq k \leq n} (S_k f)(x)$ for $n \in N$

$$\mu_n(x) = \max_{1 \leq k \leq n} \mathcal{R}[0,\ f(x) + f(\tau x) + \dots + f(\tau^{k-1} x)] \quad \text{for } n \in N.$$

Evidently μ_n is a membership function of an appropriate fuzzy subset of the set X. Moreover (due to R 7) μ_n is an $\mathcal{S}$-measurable bounded function.

Lemma 1. Let φ, ψ be functions defined on X such that $\forall x \in X$: $\varphi(x) \leq \psi(x)$. Then $(T\varphi)(x) \leq (T\psi)(x)$.

Proof. Straightforward.

Lemma 2. Let $f \in L_1(\lambda)$. Then

$$\int_X f\, d\lambda = \int_X Tf\, d\lambda .$$

Proof. See [2], [3].

Theorem 1. Let $(X, \mathcal{S}, \lambda, \tau)$ be a dynamical system, $f \in L_1(\lambda)$. Let the functions μ_n, $S_n^+ f$ have the above meaning. Then

$$\int_X \mu_n(x)\, f(x)\, d\lambda(x) + \int_X (1 - \mu_n(x))(S_n^+ f)(x)\, d\lambda(x) \geq 0$$

Proof. By the definition of $S_n^+ f$ it follows $\forall\, x \in X$: $(S_n^+ f)(x) \geq (S_k f)(x)$ for $k = 0, 1, 2, \dots, n$. Due to Lemma 1

$$\forall\, x \in X:\ (TS_n^+ f)(x) \geq (TS_k f)(x) = f(\tau x) + f(\tau^2 x) + \dots + f(\tau^k x)$$

for $k = 0, 1, 2, \dots, n$ and then

$$f(x) + (TS_n^+ f)(x) \geq f(x) + f(\tau x) + \dots + f(\tau^k x) = (S_{k+1} f)(x).$$

Hence

$$f(x) + (TS_n^+ f)(x) \geq (S_k f)(x)$$

for $k = 1, 2, \dots, n+1$ which follows

$$f(x) + (TS_n^+ f)(x) \geq \max_{1 \leq k \leq n+1} (S_k f)(x) \geq \max_{1 \leq k \leq n} (S_k f)(x)$$

for $x \in X$. Due to the definition of $\mu_n(x)$

$$[f(x) + (TS_n^+ f)(x)]\mu_n(x) \geq \mu_n(x) \max_{1 \leq k \leq n} (S_k f)(x) =$$

$$= \mu_n(x) \max_{0 \leq k \leq n} (S_k f)(x) = \mu_n(x)(S_n^+ f)(x).$$

After a short arrangement and integrating of the last inequality with using Lemma 2 we have:

$$\int_X \mu_n(x)\, f(x)\, d\lambda(x) \geq \int_X \mu_n(x) [(S_n^+ f)(x) - (TS_n^+ f)(x)]\, d\lambda(x) =$$

$$= \int_X [(S_n^+ f)(x) - (TS_n^+ f)(x)]\, d\lambda(x) -$$

$$- \int_X (1 - \mu_n(x))[(S_n^+ f)(x) - (TS_n^+ f)(x)]\, d\lambda(x) =$$

$$= \int_X (1 - \mu_n)(TS_n^+ f)\, d\lambda - \int_X (1 - \mu_n)(S_n^+ f)\, d\lambda .$$

Since $(1 - \mu_n)TS_n^+ f$ is a nonnegative function on X, the theorem is proved.

Q.E.D.

The following theorem can be assumed to be a limit case of the preceding one.

Theorem 2. Let $(X, \mathcal{S}, \lambda, \tau)$ be a dynamical system. Let $\mu(x) = \sup_{n \in N} \mathcal{R}[0,\ f(x) + f(\tau x) + \dots + f(\tau^{n-1}x)]$. Then

$$\int_X f(x)\mu(x)\ d\lambda(x) + \int_X (1 - \mu(x)) \sup_{n \in N} (S_n^+ f)(x)\ d\lambda(x) \geq 0 .$$

Proof. The above theorem is a simple consequence of Theorem 1 and of the fundamethal theorems of the theory of Lebesgue integral.

Q.E.D.

Let $\chi_p(x,y)$ be a membership function of the natural (non-fuzzy) ordering of the set of real numbers. If $\mathcal{R}(x,y) = \chi_p(x,y)$, then $\mu(x)$ is the membership function of the nonfuzzy set

$$E = \{ x \in X ;\ \exists\, n \in N:\ f(x) + f(\tau x) + \dots + f(\tau^{n-1}x) > 0 \} .$$

Evidently

$$\mu(x) = \begin{cases} 1 & \text{iff } x \in E, \text{ i.e. } \sup_{n \in N} (S_n^+ f)(x) > 0 \\ 0 & \text{iff } x \notin E, \text{ i.e. } \sup_{n \in N} (S_n^+ f)(x) \leq 0 \end{cases}$$

and then $\int_X (1 - \mu(x)) \sup_{n \in N} (S_n^+ f)(x)\ d\lambda(x) = 0$. Hence

$$\int_X f(x)\,\chi_E(x)\ d\lambda(x) = \int_E f(x)\ d\lambda(x) \geq 0 .$$

The last result is the assertion of the classical maximal ergodic theorem.

References

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