10

ON THE FUZZY APPROACH TO SOME PROBLEMS OF MEASURE THEORY

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We shall deal here with two problems. The first one is an axiomatic definition of the probability of fuzzy events. The second one is a fuzzy description of a concept of measure theory (so-called small systems); fuzzy set theory gives here an interesting simplification.

1. Probability on fuzzy events

By a fuzzy event we mean a measurable function $u_A : X \to \langle 0, 1 \rangle$, where (X, S, Q) is a probability measure space. Zadeh [1] defined the probability $P(A)$ of this event by the equality

$$P(A) = \int_X u_A \, dQ$$

(see also [2], [3], [4], [5], [6]). Now we present an axiomatic characterization.

Definition 1. Let X be a set. By a fuzzy measurable space we shall call a pair (X, F), where F is a set of fuzzy sets $u_A : X \to \langle 0, 1 \rangle$ satisfying the following properties:

1. F contains 1_X, and it is closed under proper differences ($u_A, u_B \in F$, $u_B \leqq u_A \implies u_A - u_B \in F$) and countable unions ($u_{A_n} \in F$ $(n = 1, 2, \ldots) \implies \sup_n u_{A_n} \in F$).

2. If $u_A \in F$ and $k \in R$ is such that $0 \leqq k u_A \leqq 1$, then $k u_A \in F$.

Definition 2. By a fuzzy probability measure we shall mean a mapping $P : F \to \langle 0, 1 \rangle$ defined on a fuzzy measurable space (X, F) and satisfying the following properties:

1. $P(1_X) = 1$.

2. P is fuzzy additive, i.e. if $u_A = u_{A_1} + \dots + u_{A_n}$, then $P(u_A) = \sum_{i=1}^{n} P(u_{A_i})$.

3. P is continuous, i.e. $u_{A_i} \in F$, $u_{A_i} \leq u_{A_{i+1}}$ $(i=1,2,\dots)$ implies $P(u_A) = \lim_{i\to\infty} P(u_{A_i})$.

A very simple example of an object satisfying all conditions stated in Definitions 1 and 2 is the set F of all events and the Zadeh probability P on F. We shall shaw that it is the general case.

Theorem 1. Let $P: F \to \langle 0, 1\rangle$ be a fuzzy probability measure defined on a fuzzy probability space (X, F). Then there is a (non-fuzzy) probability space (X,S,Q) such that every $u_A \in F$ is S-measurable (i.e. it is a fuzzy event) and

$$P(u_A) = \int_X u_A \, dQ \tag{1}$$

for every $u_A \in F$.

Proof. First we construct a standard probability space (X,S,Q) putting $S = \{ A \subset X ;\ \chi_A \in F \}$, $Q(A) = P(\chi_A)$, $A \in S$. Further we prove that

$$P(k\,\chi_A) = k\,P(\chi_A) \tag{2}$$

for every set $A \in S$ and every number $k \in \langle 0, 1\rangle$. The equality (2) can be proved first for rational k by fuzzy additivity of P and then for arbitrary real k by the fuzzy continuity of P.

Now we want to prove that every $u_A \in F$ is measurable with respect to (X,S). First we prove that

$$\chi_B \in F, \text{ whenever } B = \{ x \in X;\ u_A(x) \geq r \} ,\ r \in \langle 0,1\rangle \tag{3}$$

But χ_B is the pointwise limit of a sequence $(f_n)_n$ of members of F,

$$f_i = \frac{n}{r}\left[\min(u_A, r) - \min(u_A, r - \frac{r}{n}) \right] .$$

Therefore $\chi_B \in F$, since F is closed under supremums and infimums. But (3) implies now that $\{ x \in X ; u_A(x) \geq r \} \in S$ for all $r \in R$, hence u_A is a random variable. Now this fact, the fuzzy additivity of P and (2) imply that for any $u_A \in F$ of the form $u_A = \sum_{i=1}^{n} k_i \chi_{A_i}$ with A_i disjoint, we have $A_i \in S$, so u_A is a simple random variable and

$$P(u_A) = \sum_{i=1}^{n} P(k_i \chi_{A_i}) = \sum_{i=1}^{n} k_i \, P(\chi_{A_i}) = \sum_{i=1}^{n} k_i Q(A_i) =$$

$$= \int_X u_A \, dQ \quad ,$$

hence (1) holds for simple functions. But every $u_A \in F$ can be represented as a limit of a sequence (non decreasing) of simple random variables of F. Hence (1) follows by the continuity of P and the Beppo Levi theorem.

2. Small systems

In many problems of measure theory it is not necessary to know the value of $\mu(E)$ but only the fact wheather $\mu(E) = 0$ or not. In other problems one would want to know only wheather $\mu(E)$ is small or not. Of course, the property to have a small measure has a fuzzy character, so we may study a fuzzy set of small sets.

Definition 3. Let (X,S) be a measurable space. By a fuzzy set of small sets we shall call any mapping $m: S \longrightarrow \langle 0,1 \rangle$ satisfying the following conditions:

(i) $m(\emptyset) = 1$.

(ii) If $A \subset \bigcup_{i=1} A_i$

(ii) If $A \subset \bigcup_{i=1}^{\infty} A_i$, A_i, $A \in S$ $(i=1,2,\ldots)$, then $m(A) \geq \prod_{i=1}^{\infty} m(A_i)$.

(iii) If $A_i \in S$, $A_i \supset A_{i+1}$ $(i=1,2,\ldots)$ and $\bigcap_{i=1}^{\infty} A_i = \emptyset$, then $\lim_{i \to \infty} m(A_i) = 1$.

We shall show that the notion introduced in the preceding definition is actually equivalent to a quite old concept of small systems. This concept was introduced in 1966 ([7]) and has many applications (for a review see [8] and [9]) not only to the measure theory but also in the subadditive measure and integration theory, probability theory and statistics. One of the possible formalization of the concept is the following.

Definition 4. Let (X,S) be a measurable space. By a small system we understand a sequence $(N_n)_{n=1}^{\infty}$ satisfying the following properties:

(i) $\emptyset \in N_{n+1} \subset N_n \subset S$ for all $n \in N$.

(ii) If $A \in N_n$, $B \in S$ and $B \subset A$, then $B \in N_n$.

(iii) To every $n \in N$ there are $k_i \in N$ $(i=1,2,\ldots)$ such that $E_i \in N_{k_i}$ $(i=1,2,\ldots)$ imply $\bigcup_{i=1}^{\infty} E_i \in N_n$.

(iv) If $E_i \in S$, $E_i \supset E_{i+1}$ $(i=1,2,\ldots)$ and $\bigcap_{i=1}^{\infty} E_i = \emptyset$, then to every $n \in N$ there is $i \in N$ such that $E_i \in N_n$.

If we have a finite measure space (X,S,μ), then a very simple example of a small systems is the following:

$$N_n = \left\{ E \in S \,;\; \mu(E) < \frac{1}{n} \right\}.$$

Further, if we have a fuzzy set $m: S \to \langle 0, 1 \rangle$ of small sets, then a corresponding small system $(N_n)_{n=1}^{\infty}$ can be defined by the equality

$$N_n = \left\{ E \in S \,;\; m(E) > e^{-1/n} \right\}.$$

More interesting is the opposite direction.

Theorem 2. To every small system $(N_n)_n$ there exists a fuzzy set m of small sets such that $(N_n)_n$ and m are equivalent in the following sense:

(i) To every $\varepsilon > 0$ there is $n \in N$ such that $E \in N_n$ implies $m(E) > 1 - \varepsilon$.

(ii) To every $n \in N$ there is $\varepsilon > 0$ such that $m(E) > 1 - \varepsilon$ implies $E \in N_n$.

Proof. Put $h(E) = \sup \{ n; E \in N_n \}$, $f(E) = e^{-h(E)}$, $p(E) = \inf \{ \sum_{i=1}^{n} f(E_i) ; E = \bigcup_{i=1}^{n} E_i, E_i \in S, n \in N \}$, $m(E) = e^{-p(E)}$.

Then evidently $p(\emptyset) = 0$, p is σ-subadditive and upper continuous, hence m is a fuzzy set of small sets. Moreover, in [10] has been proved that

$$p(E) \leq f(E) \leq 2p(E). \tag{4}$$

Let ε be an arbitrary positive number, $\varepsilon < 1$. Choose $n \in N$, $n > - \ln(- \ln(1 - \varepsilon))$. Then $E \in N_n$ implies $h(E) \geq n$, so by (4) $p(E) \leq f(E) \leq e^{-n}$. Therefore $m(E) = e^{-p(E)} \geq \exp(-e^{-n}) > 1 - \varepsilon$. Similarly (ii) can be proved by the help of the right inequality in (4).

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