INDEXED DYNAMIC MODEL

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1. Introduction

As we well known, one of the main tasks of economical sciences is to construct models at which we first of all want to establish that they are logically possible and consistent, and next make sure that our models are realistic and therefore practical value. The extensive and fruitful investigation in this direction has started when mathematicians began to concern themselves - using modern and powerfull mathematical tools - with this part of sœience. Nevertheless for a long time both economists and mathematicians have been familiar that the classical (crisp) mathematical tools do not contribute to a better understanding of economics as much as they were expacted to. It seems now that the fuzzy subsets theory - a broadly and rapidly developing field - initiated by Zadeh (1965) provides a methodology and mathematical apparatus more adequate then the crisp ones (e.g. Pfeilsticker (1981)). So, it is not surprising that recently these new ideas are have found their expression in papers initiated a new branch of economical sciences, the fuzzy economy - as we would call it. Among these papers are papers by Ponsard (1979, 1981, 1982). Because Zadeh's theory does not allow to take specific properties of economic processes into consideration, so we will use the indexed fuzzy subset theory which was introduced by Matłoka (1984). In the section 2 we presented the basic definitions of this theory and the fundamental definitions and properties of the indexed fuzzy multi-valued functions which are necessary for our further considerations on indexed fuzzy economical systems. In the sections 3 and 4 we present an indexed fuzzy model of economic dynamics. Our own investigations on fuzzy economy differ somewhat from Ponsard's attempt. Namely we would like to initiate an investigation in fuzzy economic models under a direct influence of the monography by Makarov and Rubinov (1973). In these sections we deal with fuzzy economic systems laying special emphasis on various concepts of trajectories and their optimality.

2. Basic definitions of the indexed fuzzy sets theory and of the indexed fuzzy multi-valued functions.

Let T be a subset of real numbers R and let $G_T^{\langle 0,1\rangle}$ denotes the family of all functions $g: T \to \langle 0,1\rangle$. Let Y denotes arbitrary, but for further considerations fixed set. Next P(Y) denotes the family of all non-void subsets of Y. Let F be a mapping from T to P(Y). So, $\forall\ t \in T\ \ F(t) \subset Y$. Instead of F(t) we will write F_t.

Definition 2.1: A generalized Cartesian product of the sets F_t $(t \in T)$, F(T) say, is the set of all functions $f: T \to Y$ such that $f(t) \in F_t$, $\forall\, t \in T$.

Definition 2.2: An index fuzzy subset, v say, is a function from $G_T^{\langle 0,1\rangle}$.

The set T and the fuzzy subset v we will call the set of time-moments and the fuzzy time-moment respectively.

Definition 2.3: An indexed fuzzy subset of F(T), A_v say, is a mapping $\mu_{A_v}: F(T) \to G_T^{\langle 0,1\rangle}$ such that

(i) if $v(t)=0$ then $\mu_{A_v}(f)(t)=0$, $\forall f \in F(T)$ and $\forall t \in T$,

(ii) if there exists an element $t \in T$ such that $f'(t)=f''(t)$ then

$$\mu_{A_v}(f')(t)=\mu_{A_v}(f'')(t), \quad f', f'' \in F(T).$$

Let F(v) be a set of all functions $f \in F(T)$ such that:

– if $v(t)=0$ then $f(t)=0$.

Definition 2.4: Sendograph, $\mathrm{send}A_v$ say, of an indexed fuzzy subset A_v we call the set

$$\mathrm{send}A_v = \{(f,r): r \in G_v^{\langle 0,1\rangle},\ f \in A_v^r,\ f \in F(v)\},$$

where A_v^r denotes the r-cut of A_v (see Matłoka (1984)).

Definition 2.10: An indexed fuzzy subset A_v is called closed if its sendograph is a closed (crisp) set.

Let $P(A_v)$ denotes the family of all non-void indexed fuzzy subset of A_v. Analogously P(F(T)) stand for family of all non-void crisp subsets of F(T).

Definition 2.5: A Cartesian product, $F'(T) \times P(F''(T))$ say, is the set $\{(f,B): f \in F'(T) \text{ and } B \in P(F''(T))\}$.

Let $A_{v'}$ and $A_{v''}$ are the indexed fuzzy subsets of $F'(T)$ and $F''(T)$ respectively.

Definition 2.6: A fuzzy Cartesian product, $A_{v'} \times P(A_{v''})$ say, is a fuzzy subset of $F'(T) \times P(F''(T))$ such that $\forall f \in F'(T)$ and $\forall A'_{v''} \in P(A_{v''})$, $\forall (t',t'') \in T \times T$

$$\mu_{A_{v'} \times P(A_{v''})}(f, A'_{v''})(t',t'') = \mu_{A_{v'}}(f)(t') \wedge \sup_{g \in F''(T)} \mu_{A'_{v''}}(g)(t'').$$

Definition 2.7: An indexed fuzzy multi-valued function, $a_{v'v''}$: $A_{v'} \rightsquigarrow P(A_{v''})$ say, from an indexed fuzzy subset $A_{v'}$ into $P(A_{v''})$ is an indexed fuzzy subset of the Cartesian product $A_{v'} \times P(A_{v''})$.

For any $A'_{v'} \in P(A_{v'})$ and any $A'_{v''} \in P(A_{v''})$ we set

$$\mu_{a_{v'v''}}(A'_{v'}, A'_{v''}) = \sup_{f \in F'(T)} (\mu_{a_{v'v''}}(f, A'_{v''}) \wedge \mu_{A'_{v'}}(f)).$$

Definition 2.8: A composite, $a_{v''v'''} \circ a_{v'v''}$: $A_{v'} \rightsquigarrow P(A_{v'''})$ say, of two indexed fuzzy multi-valued functions $a_{v'v''}$: $A_{v'} \rightsquigarrow P(A_{v''})$ and $a_{v''v'''}$: $A_{v''} \rightsquigarrow P(A_{v'''})$ is an indexed fuzzy multi-valued function such that $\forall f \in F'(T)$, $\forall A'_{v'''} \in P(A_{v'''})$, $\forall (t', t''') \in T \times T$

$$\mu_{a_{v''v'''} \circ a_{v'v''}}(f, A'_{v'''})(t', t''') = \inf_{t'' \in \mathrm{supp} v''} \sup_{A'_{v''} \in P(A_{v''})}$$

$$T(\mu_{a_{v'v''}}(f, A'_{v''})(t', t''), \mu_{a_{v''v'''}}(A'_{v''}, A'_{v'''})(t'', t''')),$$

where T denotes a t-norm (compare Menger (1942)).

Definition 2.9: A converse indexed fuzzy multi-valued function, $\overleftarrow{a}_{v'v''}$ say, to an indexed fuzzy multi-valued function $a_{v'v''}$: $A_{v'} \rightsquigarrow P(A_{v''})$ is such an indexed fuzzy multi-valued function of $A_{v''} \times P(A_{v'})$ that $\forall f \in F''(T)$ and $\forall A'_{v'} \in P(A_{v'})$ and $\forall (t'', t') \in T \times T$

$$\mu_{\overleftarrow{a}_{v'v''}}(f, A'_{v'})(t'', t') = \sup_{\substack{A'_{v''} \in P(A_{v''}) \\ \{f\}_{A_{v''}} \subset A'_{v''}}} \mu_{a_{v'v''}}(A'_{v'}, A'_{v''})(t', t'').$$

Definition 2.10: It is said, that an indexed fuzzy multi-valued function, $a_{v'v''}$: $A_{v'} \rightsquigarrow P(A_{v''})$ say, has the 2-monotonous property iff $\forall A'_{v''}, A''_{v''} \in P(A_{v''})$ such that $A'_{v''} \subset A''_{v''}$ and $\forall A'_{v'} \in P(A_{v'})$

$$\mu_{a_{v'v''}}(A'_{v'}, A'_{v''}) \leq \mu_{a_{v'v''}}(A'_{v'}, A''_{v''}).$$

Definition 2.11: By the graph of an indexed fuzzy multi-valued function $a_{v'v''}$ the indexed fuzzy subset, $W_{a_{v'v''}}$ in symbol, is understood of $F'(T) \times F''(T)$ characterized by the following membership function

$$\mu_{W_{a_{v'v''}}}(f,g)(t',t'') = \sup_{A'_{v''} \in P(A_{v''})} \mu_{a_{v'v''}}(f,A'_{v''})(t',t'') \wedge$$

$$\wedge \sup_{A'_{v'} \in P(A_{v'})} \mu_{\overleftarrow{a}_{v'v''}}(g,A'_{v'})(t'',t') ,$$

$f \in F'(T)$, $g \in F''(T)$, $(t',t'') \in T \times T$.

Definition 2.12: An indexed fuzzy multi-valued function $a_{v'v''}$ is called closed iff its graph $W_{a_{v'v''}}$ is a closed indexed fuzzy subset.

For any indexed fuzzy multi-valued function $a_{v'v''}$, for any $f \in F'(T)$ and for any $(t',t'') \in T \times T$ it is denoted a fuzzy subset $W^{f(t'),t''}_{a_{v'v''}}$ such that $\forall y \in F''_{t''}$

$$\mu_{W^{f(t'),t''}_{a_{v'v''}}}(y) = \mu_{W_{a_{v'v''}}}(f,g)(t',t'') ,$$

where $g \in F''(T)$ such that $g(t'') = y$.

Let $K_{v'}$ and $K_{v''}$ denote convex indexed fuzzy cones in $F'(T)$ and $F''(T)$ respectively (see Matłoka (1984)), and let $\forall t \in T$ F'_t and F''_t are linear reference spaces.

Definition 2.13: An indexed fuzzy multi-valued function, $a_{v'v''}$: $K_{v'} \rightsquigarrow P(K_{v''})$ say, is a superlinear indexed fuzzy multi-valued function iff it is

(i) closed,

(ii) 2-monotonous,

(iii) $(0,g) \notin \operatorname{supp} W_{a_{v'v''}}$ for $g \neq 0$,

(iv) $\mu_{a_{v'v''}}(\lambda \cdot f, \lambda \cdot A_{v''}) = \mu_{a_{v'v''}}(f,A_{v''})$ $\quad \forall f \in F'(T)$,

$\forall A_{v''} \in P(K_{v''})$; $\forall \lambda > 0$,

(v) $\mu_{a_{v'v''}}(f'+f'', A'_{v''}+A'_{v''}) \geq \mu_{a_{v'v''}}(f',A'_{v''}) \wedge$

$\wedge \mu_{a_{v'v''}}(f'',A''_{v''})$ $\quad \forall f',f'' \in F'(T)$, $\forall A'_{v''},A''_{v''} \in P(K_{v''})$.

Theorem 2.1: If $a_{v'v''}: K_{v'} \rightsquigarrow P(K_{v''})$ is a superlinear indexed fuzzy multi-valued function then the graph $W_{a_{v'v''}}$ is closed and convex indexed fuzzy cone such that $(0,g) \notin \text{supp} W_{a_{v'v''}}$ for $g \neq 0$.

It is immediate consequence of the analogous theorem for the fuzzy multi-valued functions (compare Albrycht and Matłoka (1984)) .

3. Definition of indexed fuzzy model of economic dynamics.

Let E be a set of the index sets of T with at least two different elements such that:

- $v^0 \in E$, where v^0 is an initial index set,
- $\forall\, v \in E$ supp v is a compact set of T,
- $\forall\, v', v'' \in E$, if $t' \in$ supp v' and $t'' \in$ supp v'' then $t' < t''$ i.e. $v' < v''$, or if $t' \in$ supp v' and $t'' \in$ supp v'' then $t' > t''$ i.e. $v' > v''$.

Elements of E will be called fuzzy time moment and the element v^0 - initial fuzzy time moment. In practice we do not consider all time-moments but only some time-moments. If we additionally assume that an information about choice of these time-moments is given then we shall be able to say on an fuzzy time moment.

Let $\tilde{E} = \{(v', v'') \in E \times E: v' < v''\}$.

Definition 3.1: An indexed fuzzy model of economic dynamics is an object

$$\mathcal{M} = \{E, F(T), (K_v)_{v \in E}, (a_{v'v''})_{(v'v'') \in \tilde{E}}\},$$

where

- $T = \{t \in R: t \geqslant 0\}$,
- $\forall\, t \in T$, F_t denotes n_t - dimensional Euclidean space,

- $K_v \subset F(T)$ - indexed convex fuzzy cone that for any function $f \in \text{supp}\, K_v$, $f \geq 0$,
- $a_{v'v''}$ - superlinear indexed fuzzy multi-valued function,

$$a_{v'v''} : K_{v'} \rightarrow P(K_{v''}) \ .$$

It is assumed that the class $(a_{v'v''})_{(v'v'') \in \tilde{E}}$ has the following property : if $v' < v'' < v'''$ then $\forall f,h$ $F(T)$ and $\forall (t',t''')$

$$\mu_{W_{a_{v''v'''} \circ a_{v'v''}}}(f,h)(t',t''') = \inf_{t'' \in \text{supp}\, v''} \ \sup_{g \in F(T)} Tn(\mu_{W_{a_{v'v''}}}(f,g)(t',t''), \mu_{W_{a_{v''v'''}}}(g,h)(t'',t''')) \ ,$$

where Tn is a strictly t-norm (e.g. Dubois (1980)).

The indexed fuzzy cones we can interpret as the sets of all goods which are consumed or produced by the production mapping $a_{v'v''}$. The membership function $\mu_{K_{v'}}$ and $\mu_{a_{v'v''}}$ we can interpret for instance as the consumer's preferences.

Let $t \in T$ and $x_t \in F_t$. A singleton fuzzy subset of F_t generated by an element $x_t \in F_t$, $\{x_t\}$ say, is described by the following membership function

$$\mu_{\{x_t\}}(x) = \begin{cases} \alpha \in \langle 0,1 \rangle & \text{if} \quad x = x_t \ , \\ 0 & \text{otherwise} \end{cases} \qquad x \in F_t \ .$$

Definition 3.2: A fuzzy technological trajectory of m is a family $\mathcal{X} = (\{x_{t^i}\})_{t^i \in \text{supp}\, v^i, v^i \in E}$ such that

- for each x_{t^i} there exists $f^i \in F(T)$ such that

1/ $f^i(t^i) = x_{t^i}$,

2/ $\{x_{t^i}\} = \{x_{t^i}\}_{W_{a_{v^o v^i}}^{f^o(t^o), t^i}}$ for any $v^i \in E$, $v^o < v^i$,

3/ $\{x_{t^o}\} = \{x_{t^o}\}_{K_{v^o}^{t^o}}$,

4/ $\{x_{t''}\}_{W_{a_{v'v''}}^{f'(t'), t''}} \neq \emptyset$ for any $v' < v''$.

In this case the singleton $\{x_{t^i}\}$ is called the state of the trajectory $\mathcal{X}$ at the fuzzy time moment v^i, $\{x_{t^o}\}$ is the initial state of $\mathcal{X}$. It is said that the fuzzy trajectory $\mathcal{X}$ goes out $\{x\}$ if $\{x\}=\{x_{t^o}\}$ and passes through $\{x\}$ at the fuzzy time moment v^i if the state of at the fuzzy time moment is $\{x\}$.

Theorem 3.1: (of the existence of fuzzy technological trajectory). Let $v^o, v \in E$, $v^o < v$, $\{y_{t^o}\} = \{y_{t^o}\}_{K^{t^o}_{v^o}}$ and $\{y_t\} = \{y_t\}_{W^{f^o(t^o),t}_{a_{v^o v}}} \neq \emptyset$

Then there exists a fuzzy technological trajectory $\mathcal{X}$ of $\mathfrak{M}$ going out $\{y_{t^o}\}$ and passing through $\{y_t\}$ at the fuzzy time moment v.

The proof of this theorem is similar to the analogous theorem in the fuzzy model of economic dynamics (compare Matłoka (1983)).

4. Optimal trajectories of indexed fuzzy model of economic dynamics.

Now, let us additionally assume thet there exists $v^D \in E$ such that $\forall\, v \in E$, $v \leq v^D$.

Definition 4.1: A fuzzy trajectory $\mathcal{X}$ with initial state $\{x_{t^o}\}$ and terminal state $\{x_D\}$ is called μ-optimal if

$$\mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x_D) = \sup_{x \in F_D} \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x),$$

where $f^o(t^o) = x_{t^o}$.

Theorem 4.1: A fuzzy trajectory $\mathcal{X}$ with initial state $\{x_{t^o}\}$ and terminal state $\{x_D\}$ is μ-optimal iff the separation degree of the fuzzy subsets $W^{f^o(t^o),D}_{a_{v^o v^D}}$ and $\{x_D\}_{W^{f^o(t^o),D}_{a_{v^o v^D}}}$ is equal to

$$1 - \sup_{x \in F_D} \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x) .$$

Proof. Let χ denotes a μ-optimal fuzzy trajectory with initial and terminal states $\{x_{t^o}\}$ and $\{x_D\}$ respectively. Because the fuzzy multi-valued function $a_{v^o v^D}$ is superlinear so the fuzzy subsets $W^{f^o(t^o),D}_{a_{v^o v^D}}$ and $\{x_D\}_{W^{f^o(t^o),D}_{a_{v^o v^D}}}$ are bounded and convex. Therefore, with respect to their separation degree is equal to

$$1 - \sup_{x \in F_D} \left(\mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x) \wedge \mu_{\{x_D\}_{W^{f^o(t^o),D}_{a_{v^o v^D}}}}(x)\right) =$$

$$1 - \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x_D) = 1 - \sup_{x \in F_D} \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x) .$$

Let now the separation degree of two fuzzy subsets $W^{f^o(t^o),D}_{a_{v^o v^D}}$ and $\{x_D\}_{W^{f^o(t^o),D}_{a_{v^o v^D}}}$ will be equal to $1 - \sup_{x \in F_D} \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x)$.

So, in accordance with The Separations Theorem (compare Zadeh (1965)) for fuzzy subsets we have

$$\sup_{x \in F_D} \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x) = \sup_{x \in F_D} \left(\mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x) \wedge \mu_{\{x_D\}_{W^{f^o(t^o),D}_{a_{v^o v^D}}}}(x)\right) =$$

$$= \mu_{W^{f^o(t^o),D}_{a_{v^o v^D}}}(x_D) .$$

Therefore the fuzzy trajectory χ is μ-optimal.

Definition 4.2: A fuzzy trajectory χ with initial state $\{x_{t^o}\}$

and terminal state $\{x_D\}$ is called p-optimal if there exists a non-zero functional $p \in (\text{supp } K^D_{v^D})^*$ such that

$$p(x_D) = \max_{x \in \text{supp } W_{a\,v^o v^D}^{f^o(t^o),D}} p(x) > 0 ,$$

where $f^o \in F(T)$, $f^o(t^o) = x_{t^o}$.

For a normal covering of the set A we use the symbol nA (compare Makarov and Rubinov (1973)) .

Let A denote a subset of F_D, ($A \neq \emptyset$, $\{0\}$). An element x of A is called the limiting point from above of A if $ax \notin A$ for $a > 1$.

Let us assume that $\text{supp } W_{a\,v^o v^D}^{f^o(t^o),D} \neq \emptyset, \{0\}$.

Theorem 4.2: A fuzzy trajectory $\mathcal{X}$ with initial and terminal states $\{x_{t^o}\}$ and $\{x_D\}$ respectively is p-optimal iff x_D is a limiting point from above of $\text{nsupp } W_{a\,v^o v^D}^{f^o(t^o),D}$.

Proof. Let $\mathcal{X}$ denote a p-optimal fuzzy trajectory with initial and terminal states $\{x_{t^o}\}$ and $\{x_D\}$ respectively. We are going to prove that x_D is a limiting point from above of $\text{nsupp } W_{a\,v^o v^D}^{f^o(t^o),D}$.

In contrary there exists a $\lambda > 1$ such that

$$\lambda \cdot x_D \in \text{nsupp } W_{a\,v^o v^D}^{f^o(t^o),D} .$$

Because of $p(x_D) > 0$ we get

$$p(x_D) = \max_{x \in \text{supp } W_{a\,v^o v^D}^{f^o(t^o),D}} p(x) \geq p(\lambda \cdot x_D) = \lambda \cdot p(x_D) > 0$$

i.e. $1 \geq \lambda$ in contradiction with $\lambda > 1$.

Now, let us assume that x_D is a limiting point from above of the set $A = \text{nsupp}\ W_{a}^{f^o(t^o),D}{}_{v^o v^D}$. Let S denote the sphere A – A and $\|\cdot\|_A$ Minkowski's norm. It is inown that this norm is monotonous and

$$A = \{z:\ \|z\|_A \leq 1\}.$$

Because x_D is a limiting point from above of the set A there holds

$$\|x_D\|_A = 1 .$$

Therefore there exists a functional $p \in (\text{supp}\ K^D_{v^D})^*$ such that

$p(x_D) = \|x_D\|_A = 1$, $\|p\| = 1$.

So, the proof is finished.

From the above proof it follows immediately

Theorem 4.3: A fuzzy trajectory $\mathcal{X}$ with initial and terminal states $\{x_{t^o}\}$ and $\{x_D\}$ respectively is called p-optimal iff

$$\|x_D\|_{\text{nsupp}\ W_{a}^{f^o(t^o),D}{}_{v^o v^D}} = 1 .$$

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