The Possibility Image Transform and Logical Convolution

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It is now well-appreciated that an image represents a probability law on position for incoming photons.[1,2] A recent mutation of probability theory is "possibility" theory.[3] It departs from probability theory in the following basic ways: If A and B are two disjoint events, the probabilities P(A or B), P(A and B) obey

$$P(A \text{ or } B) = P(A) + P(B) \text{ and } P(A \text{ and } B) = P(A)P(B) \tag{1a}$$

while the corresponding possibilities obey

$$\Pi(A \text{ or } B) = \max(\Pi(A), \Pi(B)) \text{ and } \Pi(A \text{ and } B) \leq \min(\Pi(A), \Pi(B)). \tag{1b}$$

Operation max (a,b) = the larger of a,b, while min (a,b) = the smaller of a,b. For example, if P(A) = 0.2 and P(B) = 0.5, then P(A or B) = 0.7 while P(A and B) = 0.1; whereas if likewise $\Pi(A) = 0.2$ and $\Pi(B) = 0.5$, then $\Pi(A \text{ or } B) = 0.5$ and $\Pi(A \text{ and } B) = 0.2$. Thus, while probabilities follow algebraic operations such as multiply and add, possibilities follow logical operations such as compare and choose the largest, etc. Needless to say, these are nonlinear operations as well so that filter functions cannot be used to describe them.

As discussed with great clarity by its inventor Zadeh,[3] whereas P(A) describes the frequency of occurrence of event A, $\Pi(A)$ describes the capacity for A to happen. For example, if A_n defines the event "n eggs eaten for breakfast", for a typical person $P(A_1) = 0.2$, $P(A_2) = 0.7$, $P(A_3) = 0.09$, $P(A_4) = 0.01$, i.e., a strong peak at 2 eggs; whereas $\Pi(A_1) = 1$ (maximum value for a possibility), $\Pi(A_2) = 1$, $\Pi(A_3) = 0.8$, $\Pi(A_4) = 0.2$, $\Pi(A_5) = 0$. That is, certainly the person can equally-well eat one or two eggs, with some diminished capacity for three. Four or more eggs, however, might overwhelm the capabilities of his digestive system.

The Possibility Transform

One of the major aims of possibility theory is to produce useful solutions in situations where the probability law is unknown. But this will not be our domain of application. We assume instead that a probability law (the image) is given. From this we want to somehow construct its possibility law. What would such a picture $\{\Pi_n\}$ look like?

In fact, the job has been done. Dubois and Prade[4] have invented the following transform of a probability law $\{i_n\}$,

$$\Pi_m = \sum_{n=1}^{N} \min(i_m, i_n) \tag{2a}$$

For example if $\{i_n\} = 0.3, 0.4, 0.1, 0.2$, then $\Pi_1 = 0.3 + 0.3 + 0.1 + 0.2 = 0.9$. This defines what may be called a "possibility image transform" function $\{\Pi_m\}$. Note that it is mathematically a possibility density function. The corresponding cumulative possibility $\Pi(A)$ defining the possibility of a photon landing anywhere within an area A is defined to obey

$$\Pi(A) = \max(\Pi_m) \tag{2b}$$

where the maximization is over all pixels x_m lying within area A. Possibilities Π in (2b) obey and axioms (1b) required.* Note, by comparison with (2b), the corresponding cumulative probability of an event within area A,

$$P(A) = \sum_m i_m . \tag{3}$$

Thus, the net probability of finding a photon within an area A is the total intensity within A, while the net possibility is simply the brightest pixel within A.

Some properties of the possibility image $\{\Pi_m\}$ gleaned from (2a) are as follows. If $i_k = 0$, $\Pi_k = 0$ as well. Zeroes map into zeroes. Likewise, if i_j = maximum, $\Pi_j = 1$, the maximum possibility value allowed. Maxima map into maxima. If $i_j = i_k$, then $\Pi_j = \Pi_k$. Hence if i_j is a periodic function, so is Π_j, and with the same period. Finally, if $i_{n+1} > i_n$ likewise $\Pi_{n+1} > \Pi_n$. The possibility image has the same overall structure as the ordinary image. (However the structure is somewhat distorted, as we shall see.)

The analytic properties of the possibility image may be most easily found from the continuous transform version of (2a),

$$\Pi(y) = \int dx \min[i(y), i(x)]. \tag{4}$$

Observing Fig. 1, $\Pi(y)$ is simply the shaded area. Physically, it is as if the curve were filled with water up to the level $i(y)$. The amount of water present is then the output $\Pi(y)$.

By analytically evaluating the integral (4), it is found that a sinusoidal input $i(x) = (f/n\pi)(1 - \cos fx)$, $0 \le x \le n\pi/f$, maps into a distorted sinusoidal output

$$\Pi(y) = (fy/\pi - 1)\cos fy - \pi^{-1}\sin fy + 1 \tag{5}$$

for $0 \le y \le \pi/f$, with propagation to other y-values using symmetry about point $y = \pi/f$. See Fig. 2. Hence, the possibility transform does not decrease the modulation of the input; it merely distorts the input wave into another periodic function of the same modulation.

To test the noise propagation properties of the transform, we added uniformly random noise of amplitude 50% of the signal maximum to the input sinusoid. This net signal was then transformed via (2a). See the dashed curves in Fig. 2. The possibility image suffers less from the noise: its signal-to-noise ratio is 2.5, vs 2.1 for the input curve. In general, the possibility transform gives a small but significant amount of noise rejection. This is without any loss of modulation, but at the expense of some distortion.

An astronomical image (the Jet in M87) and its possibility transform are shown in Fig. 3. The possibility transform has brought out some extra structure in the lower half of the picture. This is because of the transform's tendency to artificially elevate mid-range intensities (the picture is a negative), as was evident in Fig. 2.

The Logical Convolution

An ordinary or arithmetic convolution has the form

$$i_m = \sum_{n=-M}^{M} o_n s_{mn} \qquad (6)$$

where $\{o_n\}$ is an object brightness distribution, $\{s_n\}$ is a point spread function of finite support 2M, and $\{i_m\}$ is the output probability image. As can be seen, for each m there are 2M multiplications and 2M additions to be performed, a total of 4M arithmetic operations. From the probabilistic viewpoint, i_m is a probability formed using the "law of total probability" (6) where o_n is the probability of a photon radiating from pixel n in the object, and s_{mn} is the probability of "spread" distance m for a photon if its object position was n. The products in (6) occur because positions m and n are independent, and the sum occurs because position m can be attained disjointly in 2M different ways. But referring to axioms (1b) for possibilities, if i, o and s were instead possibility distributions the law corresponding to (6) would be

$$i_m = \max[\min(o_{-M},s_{m,-M}),\ \min(o_{-M+1},s_{m,-M+1}),\ldots,\min(o_M,s_{mM})] \qquad (7)$$

where operation $\max(a_1,\ldots, a_M)$ = the largest a-value in the list. We call (7) a "logical convolution" because a convolution-like operation is being accomplished but by size comparisons alone. Note that 2M+1 such comparisons are required for a single output i_m, compared with 4M arithmetic operations for the arithmetic convolution (6). Also, an output i_m must be identically one of the numbers $\{o_n\}$,$\{s_n\}$ in the brackets of (7). This is a kind of closure relation which would aid in reducing the bandwidth needed for transmission of the $\{i_m\}$ as data values.

Fig. 4 shows an input object, its arithmetic convolution with the given point spread function (dashed), and its logical convolution (solid). Worth noting are that (a) where the object consists of isolated point sources, the logical convolution about equals the arithmetic convolution, but (b) where the spread functions overlap the logical convolution shows more resolution than does the arithmetic convolution. Amazingly, the logical convolution never fails to resolve two adjacent point sources, no matter how broad the point spread function is, provided there is no noise in the image data.

Because of the potential speed of its operations, the logical convolution may find a place in the transmission of image information. The "closure" property should also aid in bandwidth compression. Finally, if an optical system could be made that forms its images by logical convolution instead of arithmetic convolution, such a system would exhibit superior resolution.

References

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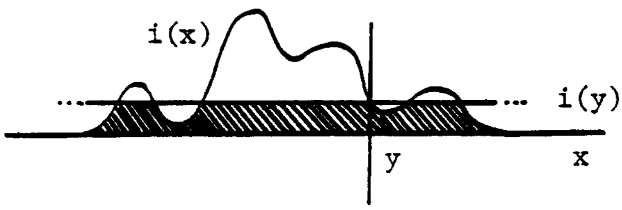

Fig. 1. Π(y) is the shaded area defined by reference level i(y).

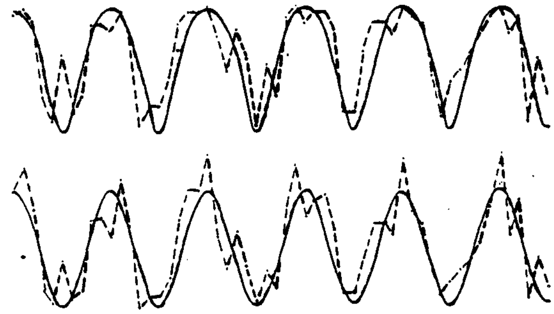

Fig. 2. Sinusoidal input (bottom, solid) and possibility output (top, solid), noisy input (bottom, dashed) and possibility output (top, dashed). Some noise suppression is apparent.

Fig. 3. (Left) The Jet (upper left) in galaxy M87 (lower right). (Right) Its possibility image. Both images are displayed with the same, linear intensity mapping function. The possibility image shows more medium-intensity details (dark in these negatives). (Photos courtesy of D. Toady, Kitt Peak National Observatory.)

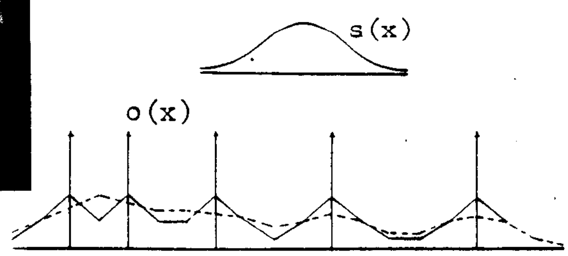

Fig. 4. Impulsive object o(x), spread function s(x), arithmetic convolution of the two (dashed), logical convolution (solid). The logical convolution has superior resolution.