ON THE CHOICE OF OPTIMAL ALTERNATIVES FOR DECISION MAKING IN PROBABILISTIC FUZZY ENVIRONMENT

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Abstract

Three main types of dominance of alternatives i.e. stochastic dominance, statistical dominance, and stochastic-statistical dominance are presented in this paper. The purpose of that is the choice of the optimal alternatives in uncertain and imprecise environment. For the decision set (as well as for the sets representing goals and constraints) the concept of probabilistic set in a sense of Hirota is applied. The considerations are illustrated by means of a numerical example.

Keywords

dominance of alternatives, decision making, probabilistic set, probabilistic fuzzy environment

1. Introduction

In some decision situations, especially in the uncertain decision making environment, the membership grade may be of probabilistic (random) nature or of possibilistic nature. The possibilistic type of uncertainty leads to fuzzy sets of type II and it is not considered here. For the membership function being a random variable for each $x \in \mathbf{X}$ the concept of probabilistic set introduced by Hirota [4] seems to be convenient. A probabilistic set A of $\mathbf{X}$ is essentially defined by the defining function

$$A: \mathbf{X} \times \Omega \rightarrow [0,1] \qquad (1.1)$$

where X represents a set of possible alternatives and A may denote decision criteria, objective functions, restrictions or goals. Ω stands for a space of simple (elementary) events.

Similarly as in fuzzy approaches to decision making problems [1,5,7,8] we can find situations, that goals and constraints constitute classes which can be described by means of the membership functions being random processes. In these situations a probabilistic fuzzy goal (constraint) is a probabilistic set of $\mathbf{X}$, characterized by its defining function. The probabilistic fuzzy decision is the probabilistic set satisfying simultaneously the goals and constraints. Denoting the respective n-ary operation on the probabilistic sets $A_1, A_2, \ldots, A_n$ by O_n, we may write for each $x \in \mathbf{X}$ as follows

$$O_n: \left\{ (A_1, A_2, \ldots, A_n) \mid A_j: \mathbf{X} \times \Omega \rightarrow [0,1] \quad j=1,2,\ldots,n \right\} \rightarrow$$
$$\rightarrow \left\{ D \mid D: \mathbf{X} \times \Omega \rightarrow [0,1] \right\} \qquad (1.2)$$

Taking into account the distribution function description of probabilistic sets [2] we have the following equality for each $x \in \mathbf{X}$

$$F_{O_n(A_1, A_2, \ldots, A_n)(x)}(z) = F_{D(x)}(z) \qquad (1.3)$$

The problem of evaluating the optimal alternatives may be solved in many ways, using various criteria. In this paper the criteria based on some types of dominance are proposed. These criteria should help for choosing the optimal alternatives in the respective decision situations.

2. Stochastic dominance

Let $Z = [0,1]$ be an attribute set and $z \in [a,b]$ be a specific level of the attribute Z, where $a<b$ and the endpoints may be 0 and 1. Let X denotes a set of all feasible alternatives and $u(z)$ be a utility function on Z. In risky decision problems, suppose that the possible impacts of two alternatives $x_i, x_j \in X$ can be described by the probability distributions $F_{D(x_i)}$ and $F_{D(x_j)}$ on Z, respectively ($D(x)$ denotes the defining function of probabilistic set D i.e. the decision set). Then the following axiom is approved under the expected utility criterion (cf. [6])

$$x_i \succcurlyeq x_j \quad \text{means} \quad E(u,\ D)(x_i) \geqslant E(u,\ D)(x_j) \tag{2.1}$$

where the symbol $\succcurlyeq$ means "preferred or indifferent to", and $E(u,\ D)(x)$ denotes mathematical expectation with respect to the utility function u and probability distribution of $D(x)$ on $[0,1]$ i.e.

$$E(u,\ D)(x) = \int_0^1 u(z)\ dF_{D(x)}(z) \tag{2.2}$$

Under the above axiom the probability distributions themselves are viewed as risky alternatives. As the available partial knowledge about the utility function the following classes of utility functions are defined

$$\begin{aligned}
\mathcal{U}_1 &= \left\{ u(z) \mid \ u \in C^1 \ , \ \frac{du}{dz} > 0 \right\} \\
\mathcal{U}_2 &= \left\{ u(z) \mid \ u \in C^2 \ ; \ u \in \mathcal{U}_1 \ , \ \frac{d^2u}{dz^2} \leqslant 0 \right\} \\
\mathcal{U}_3 &= \left\{ u(z) \mid \ u \in C^3 \quad u \in \mathcal{U}_2 \ , \ \frac{d^3u}{dz^3} \geqslant 0 \right\}
\end{aligned} \tag{2.3}$$

where C^i represents the set of bounded i-th differentiable functions. These classes are of importance for attitude of decisionmaker's preference toward risk. Obviously $\mathcal{U}_1$ is the class of utility functions for which the decisionmaker prefers an increase of the attribute level. $\mathcal{U}_2$ is the class for the decisionmaker to be risk-averse, and $\mathcal{U}_3$ is the class for the decisionmaker to be decreasingly risk-averse. With these classes the stochastic dominance is defined as follows.

For r=1,2 or 3, the distribution $F_{D(x_i)}$ dominates the distribution $F_{D(x_j)}$ in a sense of r-th degree stochastic dominance, written as

$$F_{D(x_i)} \geqslant_r F_{D(x_j)} \quad \text{if} \quad E(u, D)(x_i) \geqslant E(u, D)(x_j) \quad \text{for } u \in \mathcal{U}_r \tag{2.4}$$

The symbol $\geqslant_1$ refers to first-degree stochastic dominance, $\geqslant_2$ to second-degree stochastic dominance, and $\geqslant_3$ to third-degree stochastic dominance.

The necessary and sufficient conditions for stochastic dominance are the following ones [6].

Provided that $F_{D(x_i)}$ and $F_{D(x_j)}$ are distribution functions of a single variable

1. $F_{D(x_i)} \geqslant_1 F_{D(x_j)}$ iff $F_{D(x_j)}(z) \geqslant F_{D(x_i)}(z)$ $\quad \forall z \in [0,1]$

2. $F_{D(x_i)} \geqslant_2 F_{D(x_j)}$ iff $\int_0^z F_{D(x_j)}(t)\,dt \geqslant \int_0^z F_{D(x_i)}(t)\,dt$ $\quad \forall z \in [0,1]$

3. $F_{D(x_i)} \geqslant_3 F_{D(x_j)}$ iff $m_{F_{D(x_i)}} \geqslant m_{F_{D(x_j)}}$ and

$$\int_0^z \int_0^y F_{D(x_j)}(t)\,dt\,dy \geqslant \int_0^z \int_0^y F_{D(x_i)}(t)\,dt\,dy \quad \forall z \in [0,1] \tag{2.5}$$

where $m_{(\cdot)}$ denotes the mean value with respect to distribution function$(\cdot)$ i.e.

$$m_F = \int_0^1 z \, dF(z) \tag{2.6}$$

3. Statistical dominance

Statistical dominance of alternatives is formulated by the following components (cf. [6])

1. A decisionmaker who is faced with a choice of two alternatives, x_i, $x_j \in \mathbf{X}$, and whose criterion is the expected utility
2. A set of m mutually exclusive and exhaustive states of nature $\{N_1, N_2, \ldots, N_m\}$, whose probabilities p_k, $k \in M = \{1, 2, \ldots, m\}$, are known exactly
3. An $n \times m$ matrix of utility values in which u_{ik} is the known utility value evaluated by the decisionmaker, if he applies alternative x_i and state N_k is the true state, $i \in I = \{1, 2, \ldots, n\}$, (n denotes the total number of alternatives taken into account), $k \in M$

According to the statistical character of the consequences states, u_{ik} may be interpreted as the following two cases:

i when only a single consequence z_{ik} can possibly prevail then

$$u_{ik} = u(z_{ik}) \quad \text{for} \quad i \in I \text{ and } k \in M \tag{3.1}$$

ii when it is possible for two or more different consequences to occur then

$$u_{ik} = \int_{(z^1_{ik}, z^2_{ik}, \ldots) \in Z} u(z^1_{ik}, z^2_{ik}, \ldots) \, dF_{D(x)}(z^1_{ik}, z^2_{ik}, \ldots)$$

$$\text{for } i \in I \text{ and } k \in M \tag{3.2}$$

where $F_{D(x)}(z^1_{ik}, z^2_{ik}, \ldots)$ are the conditional probability distributions of the consequences, given the state N_k, for alternative $i \in I$

Under the above formulation, the expected utilities for alternatives x_i and x_j are written as

$$\begin{aligned} E(u, D)(x_i) &= \sum_{k=1}^{m} p_k u_{ik} \\ E(u, D)(x_j) &= \sum_{k=1}^{m} p_k u_{ik} \end{aligned} \tag{3.3}$$

respectively.

Provided that complete knowledge about probabilities of the states of nature is available, if

$$E(u, D)(x_i) \geqslant E(u, D)(x_j) \tag{3.4}$$

we say that alternative x_i dominates alternative x_j in the sense of statistical dominance.

4. Stochastic-statistical dominance

Stochastic-statistical dominance of alternatives is formulated under the following circumstances (cf. [6]):

1. The decisionmaker is faced with a selection of two alternatives, x_i and $x_j \in \mathbf{X}$, and prefers an alternative maximizing the expected utility.
2. The decisionmaker's utility function may be not identified completely but it is known which class of utility functions is preference attitude belongs to.
3. There exists a set of m mutually exclusive and exhaustive states of nature i.e. $\{N_1, N_2, \ldots, N_m\}$. One and only one of them is the true state but the decisionmaker is uncertain which N_k, $k \in M = \{1, 2, \ldots, m\}$, is the true state. Probability p_k of the state is known exactly.
4. The decisionmaker is not certain which consequence is the true one even if we knew the true state of nature. The conditional probability distributions of consequences given for each state of nature are completely assessed for each alternative.

Let $F^k_{D(x_i)}$ and $F^k_{D(x_j)}$ denote the conditional distributions respectively with respect to attribute Z. Then distributions with respect to attribute Z are given by

$$
\begin{aligned}
F_{D(x_i)}(z) &= \sum_{k=1}^{m} p_k \, F^k_{D(x_i)}(z) \\
F_{D(x_j)}(z) &= \sum_{k=1}^{m} p_k \, F^k_{D(x_j)}(z)
\end{aligned}
\qquad (4.1)
$$

Provided that complete knowledge about probabilities of the states of nature is available, if

$$E(u, D)(x_i) \geqslant E(u, D)(x_j) \qquad (4.2)$$

$$\text{for all } u \in \mathcal{U}_r \text{ and } r = 1, 2, 3$$

we say that distribution $F_{D(x_i)}$ alternative x_i dominates distribution $F_{D(x_j)}$ alternative x_j in the sense of r-th degree stochastic-statistical dominance.

5. Numerical example

Let us assume that in decision making problem the goal X is a fuzzy set for which the distribution function equals $F_X(z) = \mathbb{1}(z - x)$ (Heaviside function). The constraint given as a probabilistic set is considered as uniformly distributed. Both sets (goal and constraint) are reduced to the region $[0,1]$ as shown in Fig.1. Assuming the utility function on Z as $u(z) = z$ we get a known formula for the respective mathematical expectation[2]:

$$E(u, D)(x) = E(D)(x) = x - x^2/2 \tag{5.1}$$

where D is the decision set in a sense of minimum operation i.e.

$$D = X \cap Y_u \tag{5.2}$$

This result shows that the optimal alternative is $x_{opt} = 1$ (see Fig.1) because the maximum of $E(u, D)(x)$ is unique.
We may check independently of the explicitely assumed function $u(z) \in \mathcal{U}_1$, (where $\mathcal{U}_1$ is the class of utility functions for which the decision maker prefers an increase of the attribute level) the necessary and sufficient conditions for stochastic dominance. The respective formulas for the distribution functions and their integrals are as follows

$$F_{D(x)}(z) = \mathbb{1}(z - x)(1 - z) + z \tag{5.3}$$

$$\int_0^z F_{D(x)}(t)\,dt = \frac{z^2}{2} + \mathbb{1}(z - x)\left(z - x - \frac{z^2}{2}\right) \tag{5.4}$$

$$\int_0^z \int_0^y F_{D(x)}(t)\,dt\,dy = \frac{z^3}{6} + \mathbb{1}(z - x)\left[\frac{(z - x)^2}{2} - \frac{z^3}{6}\right] \tag{5.5}$$

These functions are illustrated in Figs. 2,3,4. They imply the respective inequalities for the first-degree stochastic dominance as well as for the second- and third-degree stochastic dominance (i.e. for r=1,2,3).
Additionally the following formula for variance is obtained under the assumption $u(z) = z$

$$V(u, D)(x) = V(D)(x) = \int_0^1 z^2\, dF_{D(x)}(z) - [E(D)(x)]^2 =$$

$$= x^2 - \frac{2}{3}x^3 - \left(x - \frac{x^2}{2}\right)^2 \tag{5.6}$$

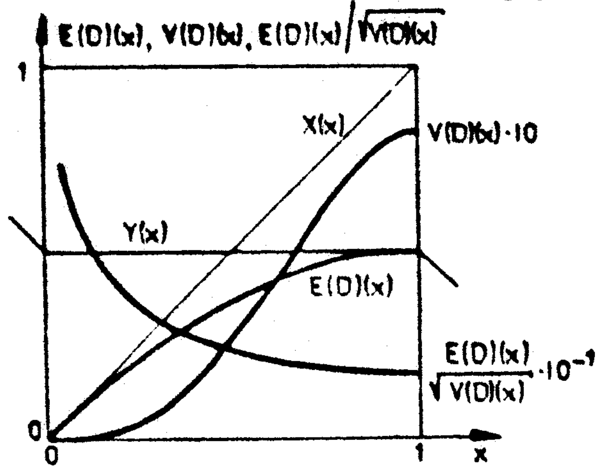

Fig.1

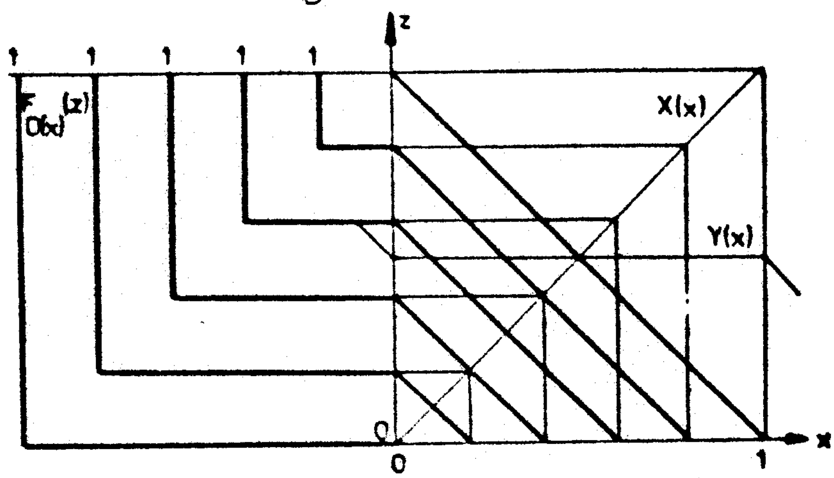

Fig.2

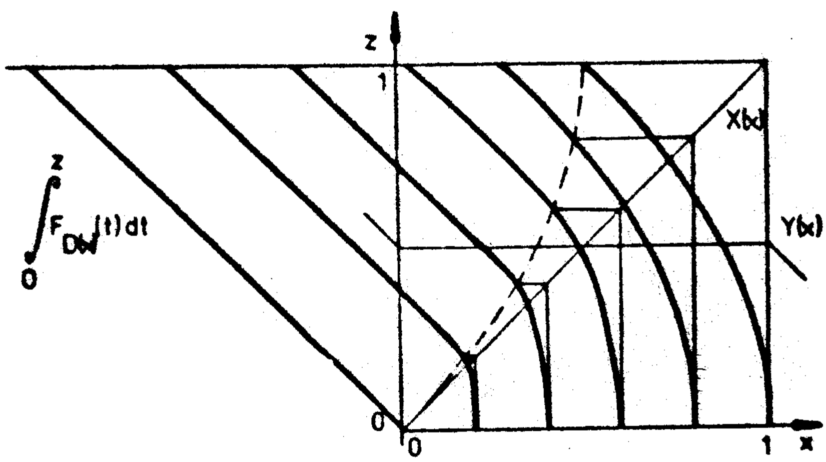

Fig.3

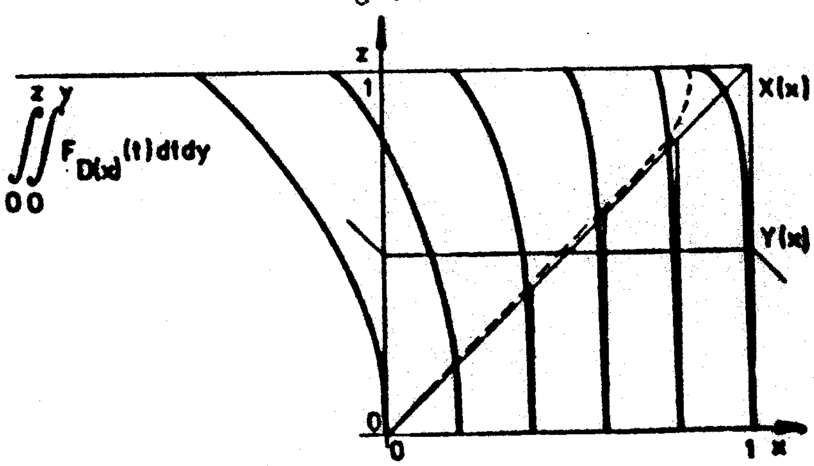

Fig.4

The respective values of variance are shown in Fig.1 as well as the values of $E(u, D)(x)/\sqrt{V(u, D)(x)}$ for each $x \in \mathcal{X}$

6. Concluding remarks

The presented types of dominance of alternatives are based on the expected utility criterion. In particular case a respective mean value may be considered.

The moment analysis of probabilistic sets shows that the main information is concentrated on lower moments such as a mean a mathematical expectation called a "membership function" of probabilistic set and a variance called a "vagueness function". In order to characterize a probabilistic set it is sufficient practically to consider both a membership function and a vagueness function.

Sometimes it may occur that the maximum of $E(u, D)(x)$ is not unique. It means that there exists a set

$$M = \{x \in \mathcal{X} \mid E(u, D)(x) = \max\} \tag{6.1}$$

In that case we need additional information using the variance For example if card $M > 1$ we have

$$x_{opt} \in \{x \in M \mid V(u, D)(x) = \min\} \tag{6.2}$$

It means that the comparison of diffrent values of variance may solve the problem of the right choice of alternatives.

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