A CONCEPT OF FUZZY GREATESTNESS AND FUZZY MAXIMALITY FOR FUZZY PREFERENCE RELATIONS

ZBIGNIEW ŚWITALSKI

Academy of Economics, Dept. of Mathematics,
Marchlewskiego 146/150, 60-967 Poznań, Poland.

Summary. In this paper we introduce a fuzzy set of greatest elements for a given fuzzy relation. Our concept is analogous to Orlovsky's idea of fuzzy nondominated alternatives and may be considered as a new approach to looking for optimal solutions in the problem of multi-criteria decision-making /when criteria are aggregated to form one fuzzy preference relation/ or in group choice theory. We study connections between FS of greatest elements and Orlovsky's FS of maximal /nondominated/ elements. Both of the FS's we define in general way, assuming that "greatestness function" has some obvious properties.

1. INTRODUCTION

If X is any set of alternatives /decisions, candidates etc./ and P any /crisp/ preference relation in X, then to find "best" or "optimal" alternatives we can use two notions: the notion of greatest element and the notion of maximal /nondominated/ element. These are defined mostly for order relations /see [1]/ but we will not assume here that P is order relation.
Element $x \in X$ is greatest for P iff $\forall y \in X \; xPy$ and maximal for P iff $\forall y \in X \; \neg yP^s x$, where $P^s = P \cap \overline{P^{-1}}$ is strict preference

relation for P. So $x \in X$ is maximal for P iff it is greatest for $\overline{(P^s)^{-1}}$. For complete P /i.e. if $P \cup P^{-1} = X \times X$ / these two notions are equivalent, because in this case $P = \overline{(P^s)^{-1}}$.
As /crisp/ preference relations are assumed mostly to be complete /see [2]/, we have in this case only one way of looking for "best" alternatives.
Situation is more complicated for fuzzy preference relations. First, there can be different definitions of P^s and hence maximality and second, we can have different notions of greatestness or maximality for fuzzy relations.

We introduce the concept of greatestness for fuzzy P in the following way. We define for every $x \in X$ "degree of greatestness" $G_P(x)$, which will be some number from the unit interval [0,1] such that if $\forall y \in X \;\; P(x,y) = 1$, then $G_P(x) = 1$ and if $\exists y \in X \;\; P(x,y) = 0$, then $G_P(x) = 0$ /P(x,y) denotes degree of preference of x over y/. We also assume that G_P fulfils some condition of monotonicity. Fuzzy set $\{(x, G_P(x))\}_{x \in X}$ will be called fuzzy set of greatest elements for P. "Best" elements we will choose from the /crisp/ set

$$G_P^* = \left\{ x \in X : G_P(x) = \bigvee_{y \in X} G_P(y) \right\} .$$

Using nondominance relation $ND(P) = \overline{(P^s)^{-1}}$ we will define also "degree of maximality" as

$$M_P(x) = G_{ND(P)}(x) .$$

Fuzzy set $\{(x, M_P(x))\}_{x \in X}$ will be called fuzzy set of maximal elements. Such FS was investigated by Orlovsky in [4] for $G_P(x) = \bigwedge_{y \in X} P(x,y)$ /denoted by $(x, \mu^{ND}(x))$ and called there

FS of nondominated alternatives/. Now "optimal" elements can be taken from the crisp set

$$M_P^* = \left\{x \in X: \ M_P(x) = \bigvee_{y\in X} M_P(y)\right\} .$$

We will study connections between G_P and M_P /and also G_P^* and M_P^* / in general case. We will prove that for relations with some completeness properties we have $G_P = M_P$ and, in some special cases, even if $G_P \neq M_P$ we can have $G_P^* = M_P^*$ /or $G_P^* \subset M_P^*$, $M_P^* \subset G_P^*$ /. So in such cases although the "degree of greatestness" and "degree of maximality" are in general different, the process of looking for optimal elements with use of these two approaches leads to the same results /this fact differs fuzzy case from the crisp one because in the last $M_P \neq G_P$ is equivalent to $M_P^* \neq G_P^*$ /.

2. NOTATION

Let X denotes non-empty set of elements. Fuzzy set /FS/ in X is a function $A: X \longrightarrow I = [0,1]$. Symbol $\mathbb{1}$ denotes universal FS in X : $\mathbb{1}(x) = 1 \quad \forall x \in X$. We define the following /crisp/ sets: $A^* = \left\{x \in X: A(x) = \bigvee_{y\in X} A(y)\right\}$, $A^1 = \left\{x \in X: A(x) = 1\right\}$, $\operatorname{supp} A = \left\{x \in X: A(x) \neq 0\right\}$. We will use also the following operations on FS's: $(\alpha A)(x) = \alpha A(x)$, $\bar{A}(x) = 1 - A(x)$, $(A \oplus B)(x) = (A(x) + B(x)) \wedge 1$, $(A \ominus B)(x) = (A(x) - B(x)) \vee 0$, $\alpha \in I$, $x \in X$. Fuzzy relation in X is a function $P: X \times X \longrightarrow [0,1]$. We define the following operations on FR's: $P^{-1}(x,y) = P(y,x)$, $(P \circ Q)(x,y) = \bigvee_{z\in X} (P(x,z) \wedge Q(z,y))$, $x,y,z \in X$. Symbol $F(X)$ stands for family of all FS's in X and $\leqslant$ denotes partial order in I or induced partial order in $F(X)$.

3. FUZZY GREATESTNESS

Let $\mathcal{P} = F(X \times X)$ be family of all FR's in X. For any $P \in \mathcal{P}$ we define mapping $e_P: X \longrightarrow F(X)$ such that $\forall x,y \in X \quad e_P(x)(y) = P(x,y)$.

<u>Definition 1.</u> Greatestness function in X is any function $G: \mathcal{P} \times X \longrightarrow I$ such that for any $P,Q \in \mathcal{P}$ and any $x,z \in X$ we have /we write $G_P(x)$ instead of $G(P,x)$/ :

/a/ $\quad e_P(x) = \mathbb{1} \Rightarrow G_P(x) = 1$,

/b/ $\quad \text{supp } e_P(x) \neq X \Rightarrow G_P(x) = 0$,

/c/ $\quad e_P(x) \leqslant e_Q(z) \Rightarrow G_P(x) \leqslant G_Q(z)$.

<u>Comment.</u> If P is crisp, then greatest elements are uniquely determined by /a/ and /b/ as: x is greatest for P iff $G_P(x) = 1$. In this case /a/ and /b/ imply /c/. In general case /c/ need not be conclusion of /a/ and /b/. Observe that property /c/ implies that $G_P(x)$ in fact depends only on the FS $e_P(x)$ because from /c/ we deduce that

$$e_P(x) = e_Q(z) \Rightarrow G_P(x) = G_Q(z) .$$

So, we have $G_P(x) = \tilde{G}(e_P(x))$ /$G_P = \tilde{G} \circ e_P$/, where $\tilde{G}$ is some /uniquely determined by G/ function $F(X) \longrightarrow I$ with properties:

/a'/ $\quad \tilde{G}(\mathbb{1}) = 1$,

/b'/ $\quad \text{supp } A \neq X \Rightarrow \tilde{G}(A) = 0$,

/c'/ $\quad A \leqslant B \Rightarrow \tilde{G}(A) \leqslant \tilde{G}(B)$.

Number $G_P(x)$ we will call degree of greatestness of element $x \in X$

with respect to the relation $P \in \mathcal{P}$. So, G_P is a fuzzy set in X and from /c/ we deduce the following monotonicity property:

$$P \leqslant Q \Rightarrow G_P \leqslant G_Q . \qquad /*/$$

Observe that /*/ is weaker then /c/.

4. FUZZY MAXIMALITY

For fuzzy relation P we define two kinds of "strict preference" relation: $P^s = P \ominus P^{-1}$ and $P^{\underset{\sim}{s}} = P \wedge \overline{P^{-1}}$ and two kinds of "nondominance" relation: $ND(P) = \overline{(P^s)^{-1}}$ and $\underset{\sim}{ND}(P) = \overline{(P^{\underset{\sim}{s}})^{-1}}$.
Using these relations we can define two kinds of fuzzy set of maximal elements:

$$M_P = G_{ND(P)},$$

$$\underset{\sim}{M}_P = G_{\underset{\sim}{ND}(P)} .$$

For crisp P we have $P \ominus P^{-1} = P \wedge \overline{P^{-1}}$ and thus $M_P = \underset{\sim}{M}_P$ = set of maximal elements for P. Relation P^s was used in [4] and $P^{\underset{\sim}{s}}$ in [3] . There can be another definitions of strict preference for fuzzy P e.g. $P^{s(T,n)} = T(P, n(P^{-1}))$, where T is some generalized conjunction and n some generalized negation. In what follows we will consider only P^s and $P^{\underset{\sim}{s}}$.

5. CONNECTIONS BETWEEN G_P AND M_P /$\underset{\sim}{M}_P$/

<u>Definition 2.</u> Let P be any FR in X. We will say that

/a/ P has property C_1 iff $P \vee P^{-1} = \mathbb{1}$;

/b/ P has property C_2 iff $\overline{P} \leqslant P^{-1}$;

/c/ P has property C_3 iff $\overline{P} = P^{-1}$.

Comment. For crisp P /a/ $\Leftrightarrow$ /b/ $\Leftrightarrow$ P is complete. In general case /a/ $\Rightarrow$ /b/ only. Property C_3 /with modifications in the set $\{(x,x)\}_{x\in X}$ / is often used as natural property for fuzzy preference relations /see e.g. [5 - 7]/. Let $\mathcal{P}_i = \{P\in\mathcal{P} : P$ has property $C_i\}$. Then $\mathcal{P}_1 \subset \mathcal{P}_2 \supset \mathcal{P}_3$.

Lemma 1. Let P be a fuzzy relation in X. Then

/a/ if $P\in\mathcal{P}$, then $P \leqslant ND(P)$;

/b/ if $P\in\mathcal{P}_1$, then $P = ND(P)$;

/c/ if $P\in\mathcal{P}_2$, then $P = \underset{\sim}{ND}(P)$;

/d/ if $P\in\mathcal{P}_2$, then $\frac{1}{2}ND(P) \leqslant P$;

/e/ if $P\in\mathcal{P}_3$, then $ND(P) = P \oplus P$.

Proof. We will prove only /a/. Proofs of /b/ - /e/ are similar. To prove /a/ it is sufficient to show that

$$P(x,y) \leqslant 1 - P^s(y,x) \qquad \forall x,y\in X .$$

It is true if $P^s(y,x) = 0$. Assume that $P^s(y,x) > 0$. Then

$$P^s(y,x) = (P(y,x) - P(x,y)) \vee 0 = P(y,x) - P(x,y).$$

Finally we obtain

$$1 - P^s(y,x) = 1 - P(y,x) + P(x,y) \geqslant P(x,y)$$

and this ends the proof of /a/.

Definition 3. Let G be a greatestness function in X. G will be called

/a/ homogeneous iff $G_{\alpha P} = \alpha G_P \qquad \forall P\in\mathcal{P} , \alpha\in I$,

/b/ distributive iff $G_{P \oplus P} = G_P \oplus G_P \quad \forall P \in \mathcal{P}$.

Theorem 1. Let G be a greatestness function. Then

/a/ if $P \in \mathcal{P}$, then $G_P \leqslant M_P$;

/b/ if $P \in \mathcal{P}_1$, then $G_P = M_P$;

/c/ if $P \in \mathcal{P}_2$, then $G_P = \underset{\sim}{M}_P$;

/d/ if G is homogeneous and $P \in \mathcal{P}_2$, then $M_P \leqslant G_P \oplus G_P$;

/e/ if G is distributive and $P \in \mathcal{P}_3$, then $M_P = G_P \oplus G_P$.

Proof. /a/, /b/ and /c/ are straightforward consequences of /a/, /b/ and /c/ of lemma 1 /we use here monotonicity condition /*//. To prove /d/ observe that /d/ from lemma 1 and homogeneity of G imply that

$$M_P = \tfrac{1}{2}M_P \oplus \tfrac{1}{2}M_P = \tfrac{1}{2}G_{ND(P)} \oplus \tfrac{1}{2}G_{ND(P)} = G_{\frac{1}{2}ND(P)} \oplus G_{\frac{1}{2}ND(P)} \leqslant G_P \oplus G_P \ .$$

/e/ is a consequence of /e/ from lemma 1 and distributivity of G.

Corrolary 1. Let G be a greatestness function in X. Then

/a/ if $P \in \mathcal{P}_1$, then $G_P^* = M_P^*$;

/b/ if $P \in \mathcal{P}_2$, then $G_P^* = \underset{\sim}{M}_P^*$;

/c/ if G is distributive, $P \in \mathcal{P}_3$, then $G_P^* \subset M_P^*$;

/d/ if G is distributive, $P \in \mathcal{P}_3$, $G_P \leqslant \frac{1}{2}$, then $G_P^* = M_P^*$;

/e/ if $P \in \mathcal{P}_3$, then $(\hat{G}_P)^* = (\hat{M}_P)^*$, where $\hat{G}_P(x) = \bigwedge_{y \in X} P(x,y)$.

Proof. /a/, /b/ and /c/ are corrolaries of /b/, /c/ and /e/ from theorem 1. /d/ is implied by /c/ because

$$G_P \leqslant \frac{1}{2} \Rightarrow G_P \oplus G_P = 2G_P, \text{ and } (2G_P)^* = G_P^* .$$

/e/ is implied by /d/ because G_P is distributive and for every $P \in \mathcal{P}_3$ $\hat{G}_P \leqslant \frac{1}{2}$.

6. A THEOREM FOR TRANSITIVE P

Definition 4. P is called transitive FR iff $P \circ P \leqslant P$.

Theorem 2. Let P be a transitive FR in X. Then

$$(\hat{M}_P)^1 \subset (\hat{G}_P)^* .$$

Proof will be given in another place.

Corrolary 2. Let X be a finite set of elements, and let P be a transitive FR in X. Then

$$(\hat{M}_P)^* \subset (\hat{G}_P)^* .$$

Proof. It can be shown /see [4]/ that for X finite and P transitive $(\hat{M}_P)^* = (\hat{M}_P)^1$. So, from theorem 2, we obtain the needed inclusion.

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