
MEASURING THE RATIONALITY OF A FUZZY PREFERENCE RELATION

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Abstract

In this paper we deal with fuzzy preference relations and its rationality, which is conceived as a fuzzy property. A measure of this "rationality" is proposed, and some results are given.

Key words: Acyclicity, Fuzzy Preference relation, Rationality.

Introduction

Let us suppose an individual who must define his preferences over a finite set X of alternatives. Such preferences may be of a fuzzy nature, and we can suppose that such an individual is able to define a "fuzzy opinion":

DEFINITION 1 :- A "Fuzzy Opinion" is a fuzzy preference relation

$$\begin{aligned} \mu : X \times X &\longrightarrow [0,1] \\ (x,y) &\longmapsto \mu(x,y) \end{aligned}$$

such that

$$\mu(x,y) + \mu(y,x) \geq 1 \quad \forall x, y \in X$$

in such a way that $\mu(x,y)$ represents the degree in which alternative x is not worse ($x \leq y$) than alternative y .

On the one hand, a Fuzzy Opinion μ can be viewed as an "outranking" relation in the sense of Roy (1977), in such a way that

$$\mu^I(x,y) + \mu^I(y,x) = 1$$

represents the degree of "indifference" ($x \perp y$) between both alternatives

$$(\mu^I(x,y) = \mu^I(y,x)) \text{ and}$$

$$\mu^S(x,y) = 1 - \mu^I(y,x)$$

represents the degree of "strict preference" ($x S y$) of alternative x over alternative y , in such a way that

$$\mu^S(x,y) + \mu^I(x,y) + \mu^S(y,x) = 1$$

On the other hand, since last property can be viewed as an orthogonality condition, a Fuzzy Opinion defines a "Fuzzy Partition" (Ruspini, 1969) of the cartesian product $X \times X$: the family of three fuzzy sets with μ^S , μ^I and μ^{-S} ($\mu^{-S}(x,y) = \mu^S(y,x) \quad \forall x, y \in X$) as their respective membership functions.

The Concept of Acyclicity

One can ask when a given fuzzy opinion can be considered as "rational". Classical works on fuzzy preference relation propose conditions like "reflexivity" ($\mu(x,x) = 1 \quad \forall x \in X$, due to Zadeh, 1971) or any type of fuzzy transitivity (see, for example, the book of Dubois-Prade, 1980). Max-min transitivity ($\mu(x,y) \geq \min\{\mu(x,z), \mu(z,y)\} \quad \forall x,y,z \in X$, proposed by Zadeh, 1971) is the usual condition of transitivity. The idea lying behind it is that the shorter the chain, the stronger the relation, in such a way that the strength of the link between two elements must be greater than or equal to the strength of any indirect chain. Though reflexivity and max-min transitivity can be justified in order to decision-making (Montero-Tejada, unpublished paper), they are not real conditions for being rational, since the set of fuzzy relations verifying each property has well-defined boundaries: intuitively we see that there are fuzzy relations with more or less rationality, so that it seems na-

natural to consider "rationality" as a fuzzy property.

One way for measuring the rationality of a fuzzy opinion, based on classical concept of "acyclicity" is the following: consider the set of alternatives $X = \{x_1, x_2, x_3\}$ and let

$$A(\{x_i\}) = \{x_i \sqcap x_i\} \quad i = 1, 2, 3$$

$$A(\{x_i, x_j\}) = \{x_i \sqsubseteq x_j, x_i \sqcap x_j, x_j \sqsubseteq x_i\} \quad \forall i \neq j$$

$$\begin{aligned} A(X) = & \left\{ x_1 \sqcap x_2 \sqcap x_3, x_1 \sqsubseteq x_2 \sqsubseteq x_3, x_1 \sqsubseteq x_2 \sqcap x_3, x_1 \sqcap x_2 \sqsubseteq x_3, \right. \\ & x_1 \sqsubseteq x_3 \sqsubseteq x_2, x_1 \sqcap x_3 \sqsubseteq x_2, x_2 \sqsubseteq x_3 \sqsubseteq x_1, \\ & x_2 \sqsubseteq x_3 \sqcap x_1, x_2 \sqcap x_3 \sqsubseteq x_1, x_2 \sqsubseteq x_1 \sqsubseteq x_3, \\ & \left. x_3 \sqsubseteq x_1 \sqsubseteq x_2, x_3 \sqsubseteq x_1 \sqcap x_2, x_3 \sqsubseteq x_2 \sqsubseteq x_1 \right\} \end{aligned}$$

be the sets of acyclic paths with groups of one, two and three alternatives. Given a fuzzy opinion defined over X , it seems natural to define the weight w of each acyclic path as follows:

$$w(x_i \sqcap x_i) = \mu^1(x_i, x_i) \quad i = 1, 2, 3$$

$$w(x_i \sqcap x_j \sqcap x_i) = \mu^R(x_i, x_j)^2 \quad \forall i \neq j$$

where R represents any basic relation \sqsubseteq , \sqcap or $\sqsubseteq \sqcap (-S)$ ($-S$, $\neg I \equiv 1$), and

$$w(x_1 \sqcap x_2 \sqcap x_3 \sqcap x_1) = \mu^{R_1}(x_i, x_j) \cdot \mu^{R_2}(x_j, x_k) \cdot \mu^{R_3}(x_k, x_i)$$

where each R_i represents the appropriate relation \sqsubseteq , \sqcap or $\neg S$, in such a way that the considered path is acyclic (for example, if $R_1 = \sqcap$ and $R_2 = \sqcap$ it must be $R_3 = \sqcap$; if $R_1 = \sqsubseteq$ and $R_2 = \sqsubseteq$, it must be $R_3 = \neg S$). Therefore, we can define

$$\Lambda_{\mu^c}(G) = \sum_{c \in A(G)} W(c) \quad \forall G \subset X$$

as a measure of acyclicity in the set G of alternatives, and

$$\Lambda(\mu^c) = \min_{G \subset X} \Lambda_{\mu^c}(G)$$

as a measure of acyclicity of the fuzzy opinion μ^c .

Now we can propose a general definition:

DEFINITION 2. Let μ be a fuzzy opinion defined over a finite set of alternatives X , and let $A(G)$ be the set of acyclic paths with lenght card (G) concerning all alternatives in $G \subset X$.

We will call "acycility" of μ to the value

$$\Lambda(\mu) = \min_{G \subset X} \Lambda_{\mu}(G)$$

where $\Lambda_{\mu}(G)$ is trivially defined as above, from

$$W(c) = \prod_i \mu^{R(c^i)}(x_i, x_{i+1}) \quad \forall c \in A(G)$$

with $R(c^i)$ the appropriate relation between x_i and x_{i+1} for the given path

$$x_1 R(c^1) x_2 R(c^2) \dots x_k R(c^k) x_1 \text{ through } G.$$

THEOREM 1. Let $G = \{x_1, \dots, x_k\}$ be a subset of alternatives. Then

$$\Lambda_{\mu^c}(G) = 1 - \left(\prod_{i=1}^k \mu(x_i, x_{i+1}) + \prod_{i=1}^k \mu(x_{i+1}, x_i) - 2 \cdot \prod_{i=1}^k \mu^1(x_i, x_{i+1}) \right)$$

with $x_{k+1} = x_1$.

Proof: on the one hand, since

$$\prod_i (\mu^S(x_i, x_{i+1}) + \mu^L(x_i, x_{i+1}) + \mu^{-S}(x_i, x_{i+1})) \leq$$

$$\sum_{e \in A(G)} \prod_i \not\models^{\mu^{R(e^i)}} (x_i, x_{i+1}) + \sum_{e \notin A(G)} \prod_i \not\models^{\mu^{R(e^i)}} (x_i, x_{i+1})$$

we get

$$A\mu^*(G) = 1 - \sum_{e \in A(G)} \prod_i \not\models^{\mu^{R(e^i)}} (x_i, x_{i+1})$$

where

$$\not\models^{\mu}(s) = \sum_{e \notin A(G)} \prod_i \not\models^{\mu^{R(e^i)}} (x_i, x_{i+1})$$

represents a measure of non-acyclicity in G .

On the other hand, it is clear that any given path

$$x_1 \xrightarrow{R(e^1)} x_2 \dots \xrightarrow{R(e^{k-1})} x_k \xrightarrow{R(e^k)} x_1$$

is in fact non-acyclic if and only if it is not the path of indifferences ($R(e^i) = 1 \forall i$) and

$$\exists R(e^i) = s, \quad \exists R(e^j) = s$$

do not hold simultaneously. Therefore,

$$A\mu^*(G) = \prod_i \not\models^{\mu(x_i, x_{i+1})} + \prod_i \not\models^{\mu(x_{i+1}, x_i)} = 2 \cdot \prod_i \not\models^{\mu^I(x_i, x_{i+1})}$$

and the theorem follows immediately.

Moreover, it follows that

$$0 \leq A\mu^*(G) \leq 1$$

In all cases, and therefore

$$0 \leq \Lambda(\mu) \leq 1$$

in such a way that from now on we can talk about "acyclicity" as a fuzzy property

$$\Lambda : F(X) \longrightarrow [0,1]$$

according to definition 2.

THEOREM 2 .- Let us suppose $\mu \in F(X)$ a non-fuzzy opinion (in other words, $\mu(x_i, x_j) \in \{0,1\} \quad \forall i, j$). Then μ is acyclic if and only if $\Lambda(\mu) = 1$.

Proof: trivial, since $\Lambda(\mu)(g) = 1$ for each $g \in X$ if and only if all paths in G are acyclic.

Moreover, we can observe that

$$W(c) \in \{0,1\} \quad \forall c \in \Lambda(G)$$

for any given acyclic non-fuzzy relation, with only one path inside $\Lambda(G)$ such that $W(c) = 1$, and $\Lambda(\mu) = 0$ when μ is non-acyclic.

Example

Let us consider $X = \{x_1, x_2, x_3\}$ and the fuzzy opinion μ defined as follows:

$$\mu(x_1, x_2) = 0.6$$

$$\mu(x_2, x_1) = 0.9$$

$$\mu(x_2, x_3) = 0.5$$

$$\mu(x_3, x_2) = 0.8$$

$$\mu(x_3, x_1) = 0.4$$

$$\mu(x_1, x_3) = 0.7$$

and $\mu(x_i, x_i) = 1 \quad \forall i = 1, 2, 3$. Hence,

$$\mu^I(x_1, x_2) = 0.6 + 0.9 - 1 = -0.5$$

$$\mu^I(x_2, x_3) = 0.5 + 0.8 - 1 = 0.3$$

$$\mu^I(x_3, x_1) = 0.4 + 0.7 - 1 = 0.1$$

and $\mu^I(x_i, x_i) = 1 \quad \forall i$, and we get

$$\Lambda(\mu^I(\{x_i\})) = 1 \quad \forall i = 1, 2, 3$$

$$\begin{aligned} \Lambda(\mu^S(\{x_1, x_2\})) &= \mu^S(x_1, x_2)^2 + \mu^I(x_1, x_2)^2 + \mu^S(x_2, x_1)^2 = \\ &= (0.6 + 0.5)^2 + 0.5^2 + (0.9 + 0.5)^2 = \end{aligned}$$

$$= 0.42$$

$$\begin{aligned} \Lambda(\mu^S(\{x_2, x_3\})) &= \mu^S(x_2, x_3)^2 + \mu^I(x_2, x_3)^2 + \mu^S(x_3, x_2)^2 = \\ &= (0.5 + 0.3)^2 + 0.3^2 + (0.8 + 0.3)^2 = \end{aligned}$$

$$= 0.38$$

$$\begin{aligned} \Lambda(\mu^S(\{x_3, x_1\})) &= \mu^S(x_3, x_1)^2 + \mu^I(x_3, x_1)^2 + \mu^S(x_1, x_3)^2 = \\ &= (0.4 + 0.1)^2 + 0.1^2 + (0.7 + 0.1)^2 = \end{aligned}$$

$$= 0.46$$

$$\begin{aligned} \Lambda(\mu^S(\{x_1, x_2, x_3\})) &= 1 - \left\{ 0.6, 0.5, 0.4, 0.9, 0.8, 0.7, \right. \\ &\quad \left. - 2, 0.5, 0.3, 0.1 \right\} = 0.406 \end{aligned}$$

and therefore,

$$\Lambda(\mu^I) = 0.38$$

which means that the pair $\{x_2, x_3\}$ is the group of alternatives with the lowest acyclicity.

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