ON CONTENTS OF REALIZABLE FUZZY MATRICES

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Abstract

In this paper, we introduced the concept of D-M fuzzy matrix and given methods for constructing its realization, and showed its content $r(B_n)=n$. Finally, we studied contents of realizable fuzzy matrices B_n and showed

$$r(B_n)\le\frac{(n-1)(n-2)}{2}+1 \qquad (n\ge 4)$$

Keywords: Fuzzy matrix, Contents of fuzzy matrices, Realizable matrix.

In this paper, we used some notations and results in [1], [2], for brief, these notations and results are not list. Please refer to [1], [2].

1 D-M matrix and its content

Definition: A matrix $B_n=(b_{ij})\in L^{n\times n}$ is called D-M matrix if it satisfies following conditions:

(1) $b_{ij}=b_{ji}$, $i,j=1,2,\dots n$,

(2) $b_{ij}<b_{ik}$, $i\le k<j$, $i=1,2,\dots n$,

(3) $b_{ij}<b_{kj}$, $i<k\le j$, $j=1,2,\dots n$.

Theorem 1: Suppose that

$$B_n=\begin{pmatrix} b_{11} & b_{12} & \dots b_{1n} \\ b_{21} & b_{22} & \dots b_{2n} \\ \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots b_{nn} \end{pmatrix}$$

is a D-M matrix, then $r(B_n)=n$ and

$$A=\begin{pmatrix} b_{11} & b_{12} & \dots b_{1n} \\ & b_{22} & \dots b_{2n} \\ & 0 \ \ddots & \vdots \\ & & b_{nn} \end{pmatrix}$$

is a realization of B_n .

Proof: Because B_n is a D-M matrix, so B_n is realizable. Write $AA'=C=(c_{ij})$.

If $i\leq j$:

$$\begin{aligned} c_{ij} &= \vee\{a_{ik}\wedge a_{jk} \mid k=1,2,\ldots,n\} \\ &= \vee\{\{a_{ik}\wedge a_{jk} \mid k<j\}\cup\{a_{ik}\wedge a_{jk} \mid k\geq j\}\} \\ &= \vee\{\{a_{ik}\wedge 0\}\cup\{a_{ik}\wedge a_{jk} \mid k\geq j\}\} \\ &= \vee\{a_{ik}\wedge a_{jk} \mid k\geq j\} \\ &= \vee\{b_{ik}\wedge b_{jk} \mid k\geq j\} \\ &= \vee\{b_{ik} \mid k\geq j\} \\ &= b_{ij} . \end{aligned}$$

If $i>j$:

$$\begin{aligned} c_{ij} &= \vee\{a_{ik}\wedge a_{jk} \mid k=1,2,\ldots n\} \\ &= \vee\{\{a_{ik}\wedge a_{jk} \mid k<i\}\cup\{a_{ik}\wedge a_{jk} \mid k\geq i\}\} \\ &= \vee\{\{0\wedge a_{jk} \mid k<i\}\cup\{a_{ik}\wedge a_{jk} \mid k\geq i\}\} \\ &= \vee\{a_{ik}\wedge a_{jk} \mid k\geq i\} \\ &= \vee\{b_{ik}\wedge b_{jk} \mid k\geq i\} \\ &= \vee\{b_{jk} \mid k\geq i\} \\ &= b_{ji} \\ &= b_{ij} . \end{aligned}$$

Thus $C=AA'=B_n$, therefor A is a realization of B_n. Since A only has n columns, $r(B_n)\leq n$. From paper [2], we have $r(B_n)\geq n$, therefor $r(B_n)=n$.

2 Estimation of upper bound of contents of realizable fuzzy symmetric matrices

Proposition 1: If $B^{(i)}\in L^{n\times n}$ $(i=1,2,\ldots m)$ are realizable L-fuzzy symmetric matrices and $A^{(i)}$ is a realization of $B^{(i)}$, then

$B_n=B^{(1)}+B^{(2)}+\ldots+B^{(m)}$ is also realizable and $(A^{(1)}\ A^{(2)}\ \ldots A^{(m)})$ is a realization of B_n, and $r(B_n)=r(B^{(1)}+B^{(2)}+\ldots+B^{(m)})\leq r(B^{(1)})+r(B^{(2)})+\ldots+r(B^{(m)})$.

Proposition 2: If A is a realization of B_n , then number of different nonzero elements of A is necessarily more than or equal to number of different nonzero elements of B_n .

Proposition 3: If B_n is realizable, the b_{ij} occurs necessarily in i-th row or j-th row of its realization. If b_{ij} occurs neither in i-th row of **A** nor in j-th row of A, the A is not realization of B_n.

Above propositions are obvious.

Proposition 4: Let

$$B_n=\begin{bmatrix} b_{11} & \ldots\ldots & b_{1n} \\ b_{21} & \ldots\ldots & b_{2n} \\ & \ldots\ldots\ldots & \\ b_{n1} & \ldots\ldots & b_{nn} \end{bmatrix}$$ be a realizable L-fuzzy symmetric

matrix, then

$$A_n=\left[\begin{array}{cccccc} \underbrace{\begin{matrix} b_{11} & b_{11} & b_{11} & \ldots & b_{11} \\ b_{12} & 0 & 0 & \ldots & 0 \\ 0 & b_{13} & 0 & \ldots & 0 \\ 0 & 0 & b_{14} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & & \ldots\ldots & & \\ 0 & 0 & \ldots & b_{1n-1} & 0 \\ 0 & 0 & \ldots & 0 & b_{1n} \end{matrix}}_{A^{(1)}} & \underbrace{\begin{matrix} 0 & 0 & 0 & \ldots & 0 \\ b_{22} & b_{22} & b_{22} & \ldots & b_{22} \\ b_{23} & 0 & 0 & \ldots & 0 \\ 0 & b_{24} & 0 & \ldots & 0 \\ 0 & 0 & b_{25} & \ldots & 0 \\ & & \ldots\ldots & & \\ 0 & 0 & \ldots & b_{2n-1} & 0 \\ 0 & 0 & \ldots & 0 & b_{2n} \end{matrix}}_{A^{(2)}} & \underbrace{\begin{matrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ b_{33} & b_{33} & b_{33} & \ldots & b_{33} \\ b_{34} & 0 & 0 & \ldots & 0 \\ 0 & b_{35} & 0 & \ldots & 0 \\ & & \ldots\ldots & & \\ 0 & 0 & \ldots & b_{3n-1} & 0 \\ 0 & 0 & \ldots & 0 & b_{3n} \end{matrix}}_{A^{(3)}} & \begin{matrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{matrix} & \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ b_{n-1\,n-1} \\ b_{n-1\,n} \end{matrix}}_{A^{(n-1)}} & \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ 0 \\ b_{nn} \end{matrix}}_{A^{(n)}} \end{array}\right]$$

$$=(A^{(1)}\ A^{(2)}\ A^{(3)}\ \ldots\ldots A^{(n-1)}\ A^{(n)})$$

is a realization of B_n, and $r(B_n)\leq\frac{n(n-1)}{2}+1$

Proof: Let

$$B^{(i)}=\begin{bmatrix} 0\ldots0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0\ldots0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0\ldots0 & b_{ii} & b_{i\,i+1} & b_{i\,i+2} & \ldots & b_{i\,n-1} & b_{in} \\ 0\ldots0 & b_{i+1\,i} & b_{i\,i+1} & 0 & \ldots & 0 & 0 \\ 0\ldots0 & b_{i+2\,i} & 0 & b_{i\,i+2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0\ldots0 & b_{n-1\,i} & 0 & 0 & \ldots & b_{i\,n-1} & 0 \\ 0\ldots0 & b_{ni} & 0 & 0 & \ldots & 0 & b_{in} \end{bmatrix} \qquad (i=1,2,\ldots n)$$

then $B_n=B^{(1)}+B^{(2)}+\ldots+B^{(i)}+\ldots+B^{(n)}$ and $A^{(i)}$ is a realization of $B^{(i)}$, $r(B^{(i)})\leq n-i$ $(i=1,2,\ldots,n-1)$, $r(B^{(n)})=1$. Thus

$$\begin{aligned} r(B_n) &= r(B^{(1)}+B^{(2)}+\ldots+B^{(n)}) \leq r(B^{(1)})+\ldots+r(B^{(n)}) \\ &= (n-1)+(n-2)+\ldots+2+1+1 \\ &= \frac{n(n-1)}{2}+1 \quad . \end{aligned}$$

Proposition 5: Suppose that

$$B_n = \begin{bmatrix} b_{11} & b_{12} & \ldots b_{1n} \\ b_{21} & b_{22} & \ldots b_{2n} \\ & \ldots\ldots\ldots & \\ b_{n1} & b_{n2} & \ldots b_{nn} \end{bmatrix}$$ is a diagonally dominant matrix.

If

$$B_{n+1} = \begin{bmatrix} b_{11} & b_{12} & \ldots b_{1n} & b_{1\,n+1} \\ b_{21} & b_{22} & \ldots b_{2n} & b_{2\,n+1} \\ & \ldots\ldots\ldots & & \vdots \\ b_{n1} & b_{n2} & \ldots b_{nn} & b_{n\,n+1} \\ b_{n+1\,1} & b_{n+1\,2} & \ldots b_{n+1\,n} & b_{n+1\,n+1} \end{bmatrix} = \begin{bmatrix} & & & b_{1\,n+1} \\ & B_n & & b_{2\,n+1} \\ & & & \vdots \\ & & & b_{n\,n+1} \\ b_{n+1\,1} & b_{n+1\,2} & \ldots b_{n+1\,n} & b_{n+1\,n+1} \end{bmatrix}$$

satisfies following conditions:

for $i,j \in \{1,2,\ldots n\}$

(1) $b_{i\,n+1} < b_{ij}$,

(2) $b_{n+1\,n+1} > b_{i\,n+1}$,

(3) $b_{i\,n+1} \leq b_{j\,n+1}$ $\quad i<j$,

then $r(B_{n+1}) \leq r(B_n)+1$.

Proof: Obviously, B_{n+1} is realizable. By using the proposition 4, we know that

$A_{n+1} = (A^{(1)}\ A^{(2)}\ \ldots\ldots A^{(n)}\ A^{(n+1)})$

$$= \left[\begin{array}{ccccc|ccccc|c|cc|c|c} b_{11} & b_{11} & \ldots & b_{11} & b_{11} & 0 & 0 & \ldots & 0 & 0 & & 0 & 0 & 0 & 0 \\ b_{12} & 0 & \ldots & 0 & 0 & b_{22} & b_{22} & \ldots & b_{22} & b_{22} & \ldots\ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{13} & \ldots & 0 & 0 & 0 & b_{23} & \ldots & 0 & 0 & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ldots\ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & b_{n-1\,n-1} & b_{n-1\,n-1} & 0 & 0 \\ 0 & 0 & \ldots & b_{1n} & 0 & 0 & 0 & \ldots & b_{2n} & 0 & \ldots\ldots & b_{n-1\,n} & 0 & b_{nn} & 0 \\ 0 & 0 & \ldots & 0 & b_{1\,n+1} & 0 & 0 & \ldots & 0 & b_{2\,n+1} & & 0 & b_{n-1\,n+1} & b_{n\,n+1} & b_{n+1\,n+1} \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{A^{(1)}} \quad \underbrace{\qquad\qquad}_{A^{(2)}} \quad \ldots \quad \underbrace{\qquad\qquad}_{A^{(n-1)}} \quad A^{(n)} \quad A^{(n+1)}$$

is a realization of B_{n+1} .

Because $b_{1\,n+1} \wedge b_{11} = b_{1\,n+1} \wedge b_{n+1\,n+1}$, the matrix obtained by changing 0 of first row of last bolumn of A_{n+1} for $b_{1\,n+1}$ and canceling last bolumn of $A^{(1)}$ is also a realization of B_{n+1} [2].

Because $b_{1\,n+1} \wedge b_{2\,n+1} < b_{11} \wedge b_{12}$, $b_{2\,n+1} \wedge b_{n+1\,n+1} = b_{22} \wedge b_{2\,n+1}$, the matrix obtained by changing 0 of second row of last bolumn of A_{n+1} for $b_{2\,n+1}$ and canceling last bolumn of $A^{(2)}$ is also realization of B_{n+1}. We go on doing similarly.

Finally, we change 0 of n-th row of last bolumn of A_{n+1} for $b_{n\,n+1}$, but we can not cancele only bolumn of $A^{(n)}$. (see proposition 3). The matrix obtained by above changing is written A^*_{n+1}, then A^*_{n+1} is a realization of B_{n+1}.

$$A^*_{n+1}=\begin{pmatrix}
b_{11}\ b_{11}\ b_{11} \dots b_{11} & 0\ 0\ 0 \dots 0 & 0\ 0\ 0 \dots 0 & \dots & 0 & 0 & b_{1\,n+1}\\
b_{12}\ 0\ 0 \dots 0 & b_{22}\ b_{22}\ b_{22} \dots b_{22} & 0\ 0\ 0 \dots 0 & \dots & 0 & 0 & b_{2\,n+1}\\
0\ b_{13}\ 0 \dots 0 & b_{23}\ 0\ 0 \dots 0 & b_{33}\ b_{33}\ b_{33} \dots b_{33} & \dots & 0 & 0 & b_{3\,n+1}\\
0\ 0\ b_{14} \dots 0 & 0\ b_{24}\ 0 \dots 0 & b_{34}\ 0\ 0 \dots 0 & \dots & 0 & 0 & b_{4\,n+1}\\
0\ 0\ 0 \dots 0 & 0\ 0\ b_{25} \dots 0 & 0\ b_{35}\ 0 \dots 0 & \dots & 0 & 0 & b_{5\,n+1}\\
\dots\dots & \dots\dots & \dots\dots & \dots & . & . & .\\
0\ 0 \dots b_{1\,n-1}\ 0 & 0\ 0 \dots b_{2\,n-1}\ 0 & 0\ 0 \dots b_{3\,n-1}\ 0 & \dots & b_{n-1\,n-1} & 0 & b_{n-1\,n+1}\\
0\ 0 \dots 0\ b_{1n} & 0\ 0 \dots 0\ b_{2n} & 0\ 0 \dots 0\ b_{3n} & \dots & b_{n-1\,n} & b_{nn} & b_{n\,n+1}\\
0\ 0 \dots 0\ 0 & 0\ 0 \dots 0\ 0 & 0\ 0 \dots 0\ 0 & \dots & 0 & 0 & b_{n+1\,n+1}
\end{pmatrix}$$

$$A^*_{n+1}=\begin{pmatrix}
 & & & & & & b_{1\,n+1}\\
 & & & & & & b_{2\,n+1}\\
 & & & A_n & & & \vdots\\
 & & & & & & b_{n\,n+1}\\
0\ 0 \dots 0\ 0 & 0\ 0 \dots 0\ 0 & 0\ 0 \dots 0\ 0 & \dots & 0 & 0 & b_{n+1\,n+1}
\end{pmatrix}$$

where A_n is a realization of B_n. Therefor

$$r(B_{n+1}) \le r(B_n)+1 .$$

Theorem 2: Suppose that $B_n \in L^{n\times n}$ is a realizable fuzzy symmetric matrix, then

$$r(B_n) \le \frac{(n-1)(n-2)}{2}+1 , \quad \text{for} \quad n \ge 4 .$$

Proof: we prove it by inductive method.

If n=4. then

$$r(B_4) \le 4 = \frac{(4-1)(4-2)}{2}+1 \qquad \text{(see [2])}.$$

Assume that

$$r(B_k) \le \frac{(k-1)(k-2)}{2}+1 \quad \text{is true for } n=k.$$

If n=k+1 : Let

$$B_{k+1}=\begin{bmatrix} b_{11} & b_{12} & \dots b_{1k} & b_{1\,k+1} \\ b_{21} & b_{22} & \dots b_{2k} & b_{2\,k+1} \\ \dots\dots & \dots\dots & \dots\dots & \\ b_{k1} & b_{k2} & \dots b_{kk} & b_{k\,k+1} \\ b_{k+1\,1} & b_{k+1\,2} & \dots b_{k+1\,k} & b_{k+1\,k+1} \end{bmatrix}$$

$$=\begin{bmatrix} b_{11} & b_{12} & \dots b_{1k} & 0 \\ b_{21} & b_{22} & \dots b_{2k} & 0 \\ \dots\dots & \dots\dots & \dots\dots & \\ b_{k1} & b_{k2} & \dots b_{kk} & 0 \\ 0 & 0 & \dots 0 & 0 \end{bmatrix}+\begin{bmatrix} b_{1\,k+1} & 0 & 0 & \dots 0 & b_{1\,k+1} \\ 0 & b_{2\,k+1} & 0 & \dots 0 & b_{2\,k+1} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \\ 0 & 0 & 0 & \dots b_{k\,k+1} & b_{k\,k+1} \\ b_{k+1\,1} & b_{k+1\,2} & b_{k+1\,3} & \dots b_{k+1\,k} & b_{k+1\,k+1} \end{bmatrix}$$

$$=B^{(1)}+B^{(2)}\ .$$

Write

$$B_k=\begin{bmatrix} b_{11} & b_{12} & \dots b_{1k} \\ b_{21} & b_{22} & \dots b_{2k} \\ \dots\dots & \dots\dots & \dots\dots \\ b_{k1} & b_{k2} & \dots b_{kk} \end{bmatrix}$$ and assume that a realization

of B_k is

$$A_k=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1p} \\ a_{21} & a_{22} & \dots a_{2p} \\ \dots\dots & \dots\dots & \dots\dots \\ a_{k1} & a_{k2} & \dots a_{kp} \end{bmatrix}$$ where $p=\frac{(k-1)(k-2)}{2}+1$.

Obviously,

$$A^{(1)}=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1p} \\ a_{21} & a_{22} & \dots a_{2p} \\ \dots\dots & \dots\dots & \dots\dots \\ a_{k1} & a_{k2} & \dots a_{kp} \\ 0 & 0 & \dots 0 \end{bmatrix}$$ is a realization of $B^{(1)}$.

Write

$$A^{(2)}=\begin{bmatrix} b_{1\,k+1} & 0 & 0 & \dots 0 \\ 0 & b_{2\,k+1} & 0 & \dots 0 \\ 0 & 0 & b_{3\,k+1} & \dots 0 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ 0 & 0 & 0 & \dots b_{k\,k+1} \\ b_{k+1\,k+1} & b_{k+1\,k+1} & b_{k+1\,k+1} & \dots b_{k+1\,k+1} \end{bmatrix}$$

Obviously, $A^{(2)}$ is a realization of $B^{(2)}$.

Since $B_{k+1}=B^{(1)}+B^{(2)}$, $A_{k+1}=(A^{(1)}\ \ A^{(2)})$ is a realization of B_{k+1} , where

$$A_{k+1}=\begin{pmatrix} a_{11} & \cdots\cdots & a_{1p} & b_{1\,k+1} & \cdots\cdots & 0 & \cdots\cdots & 0 & \cdots\cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \cdots\cdots & b_{i\,k+1} & \cdots\cdots & 0 & \cdots\cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \cdots\cdots & 0 & \cdots\cdots & b_{j\,k+1} & \cdots\cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{k1} & \cdots\cdots & a_{kp} & 0 & \cdots\cdots & 0 & \cdots\cdots & 0 & \cdots\cdots & b_{k\,k+1} \\ 0 & \cdots\cdots & 0 & b_{k+1\,k+1} & \cdots\cdots & b_{k+1\,k+1} & \cdots\cdots & b_{k+1\,k+1} & \cdots\cdots & b_{k+1\,k+1} \end{pmatrix}$$

$$\underbrace{\qquad\qquad}_{A^{(1)}}\qquad\qquad\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{A^{(2)}}$$

has k+1 rows and p+k bolumns. We discuss by two cases:

(1) If there are two rows in $A^{(1)}$, such as i-th row and j-th row ($i\neq j$), satisfying $a_{ih}\wedge a_{jh}\geqslant b_{i\,k+1}\wedge b_{j\,k+1}$ for existing a $h\in\{1,2,\ldots p\}$, we may change 0 of j-th row of i-th bolumn in $A^{(2)}$ for $b_{j\,k+1}$,and cancele j-th bolumn in $A^{(2)}$. the matrix obtained by above changing is written $A^{(2)*}$. Write $A^{*}_{k+1}=(A^{(1)}\ \ A^{(2)*})$, therefor A^{*}_{k+1} is also a realization of B_{k+1} . The number of bolumns of A^{*}_{k+1} is

$$p+(k-1)=\frac{(k-1)(k-2)}{2}+1+(k-1)=\frac{(k-1)(k-2)}{2}+k$$

$$=\frac{(k-1)k}{2}+1=\frac{((k+1)-1)((k+1)-2)}{2}+1$$

thus

$$r(B_{k+1})\leq\frac{((k+1)-1)((k+1)-2)}{2}+1 \quad .$$

(2) If for $\forall\ r,h\in\{1,2,\ldots k\}$ $(r\neq h)$

$$a_{rj}\wedge a_{hj}<b_{r\,k+1}\wedge b_{h\,k+1} \quad \text{for } j=1,2,\ldots p \ ,$$

we can prove that

$$r(B_{k+1})\leq\frac{((k+1)-1)((k+1)-2)}{2}+1$$

is also hold. (The proof is omitted) Thus

$$r(B_n)\leq\frac{(n-1)(n-2)}{2}+1$$

is hold for n=k+1, therefor

$$r(B_n)\leq\frac{(n-1)(n-2)}{2}+1$$

is hold for $n \geqslant 4$.

References

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