Fuzzy relation equation on infinite sets

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Abstract

The largest element of the solution set of a given fuzzy relation equation has been found by E. Sanchez [1], but the smallest element dose not exist. In the case of the determinate relation equations, complete consequences has been found by Wang-Pei zhuang and Yuan-Meng [2]. In the case of the fuzzy relation equations, Wang and Yuan have given a necessary and sufficient condition such that the fuzzy relation equation is solvable. In the paper[3], the reachable solution set of a fuzzy relation equation is given.

In this paper, we shall give the solution set of a fuzzy relation equation on the infinite sets.

Keywords: Fuzzy Relation Equation, Fuzzy Relation Inequality Largest Solution, Reachable Solution, Quasi-minimum

A. A necessary and sufficient condition

Let $F(U\times V)=\left\{R \,\middle|\, R: U\times V \xrightarrow{g} L\right\}$ $L=\left\{x \,\middle|\, 0\leqslant x\leqslant 1\right\}$. $R\in F(U\times V)$ is called a fuzzy relation from U into **V**.

Given a fuzzy relation equation

$$X\circ R=S \qquad (\mathrm{I})$$

where $R \in F(V \times W)$, $S \in F(U \times W)$ are given and $X \in F(U \times V)$ is unknown.

$X \in F(U \times V)$ is called a solution of the fuzzy relation equation $XoR = S$ if

$$\bigvee_{v} (X(u,v) \wedge R(v,w)) = S(u,w) \text{ for } \forall (u,w). \qquad (\bigvee = \sup,\ \bigwedge = \inf)$$

Theorem 1. A neccessary and sufficient condition that $XoR = S$ be solvable is that

$$\bigvee_{v} \Big\{ \Big[\bigwedge_{S(u,w') < R(v,w')} (s(u,w')) \Big] \wedge R(v,w) \Big\} = S(u,w) \text{ for } \forall (u,w).$$

If $XoR = S$ be solvable, then the largest element of the solution set is as follows:

$$\bar{X}(u,v) = \bigwedge_{S(u,w') < R(v,w')} (S(u,w')) \quad \text{for } \forall (u,v)$$

(In this paper, we assume that the infimum of the empty set is 1)

B. Reachable solution set of $XoR \geq S$

Given a fuzzy relation inequality

$$XoR \geq S \qquad (II)$$

Definition 1. $X \in F(U \times V)$ is called the reachable solution of $XoR \geq S$ if

For $\forall (u,w)$, $\exists v^*$ such that $X(u,v^*) \wedge R(v^*,w) \geq S(u,w)$.

By analogy with the fuzzy relation equation $XoR = S$ [3], we have

Theorem 2. $XoR \geq S$ has the reachable solutions if and only if

$G = \{ g \mid U \times W \xrightarrow{g} V, \text{ for } \forall (u,w), R(g(u,w), w) \geq S(u,w) \} \neq \phi$, If $G \neq \phi$, for $\forall g \in G$ let

$$Xg(u,v) = \bigvee_{g(u,w')=v} (s(u,w')) \quad \text{for } \forall (u,v)$$

(In this paper, we assume that the supremum of the empty set is o).

Then the reachable solution set of $XoR \supseteq S$ is

$$\mathscr{X} = \left\{X \middle| X \geq Xg,\ g \in G\right\} = \bigcup_{g \in G} \left\{X \middle| X \geq Xg\right\}$$

C. Solution set of $XoR \supseteq S$

Defintion 2. For $\forall a,b \in L$, we define

$$a \dot{-} b = \begin{cases} a-b & \text{when } a > b \\ o & \text{when } a \leq b \end{cases}$$

Evidently it satisfies $a \dot{-} b \in L$ for $\forall a,b \in L$.

Lemma 1. Suppose $S_t \in L$ $t \in T$, $\mathcal{E}_n = \frac{1}{10^n}$ $(n=1,2,\ldots)$.
Then $\beta = \bigvee_t (s_t)$ if and only if β satisfies

1) for $\forall t \in T$, $S_t \leqslant \beta$

2) for $\forall \mathcal{E}_n$, $\exists t_n \in T$ such that $S_{t_n} \geqslant \beta \dot{-} \mathcal{E}_n$.

Now we consider the fuzzy relation inequalities

$$XoR \supseteq S^{(n)} \qquad (n=1,2,\ldots) \qquad \text{(III)}$$

Where

$$S^{(n)}(u,w) = s(u,w) \dot{-} \mathcal{E}_n \quad \text{for } \forall (u,w),$$

By theorem 2, the reachable solution sets are

$$\mathscr{X}^{(n)} = \left\{X \middle| X \geq Xg_n^{(n)},\ g_n \in G_n\right\} = \bigcup_{g_n \in G_n} \left\{X \middle| X \geq Xg_n^{(n)}\right\} \qquad n=1,2,\cdots$$

where

$$G_n = \left\{ g_n \middle| U \times W \xrightarrow{g_n} V, \quad \text{for } \forall (u,w),\ R(g_n(u,w),w) \geqslant S(u,w) \dot{-} \mathcal{E}_n \right\}$$

$$Xg_n^{(n)}(u,v) = \bigvee_{g_n(u,w')=v} (S(u,w') \dot{-} \mathcal{E}_n) \quad \text{for } \forall (u,v).$$

Theorem 3. If $XoR \gtrsim S$ be solvable, then the solution set is

$$\underline{\mathcal{X}} = \bigcap_{n=1}^{\infty} \{\mathcal{X}^{(n)}\} = \bigcap_{n=1}^{\infty} \bigcup_{g_n \in G_n} \{X \mid X \geq Xg_n^{(n)}\}$$

Proof. a) Suppose that X is any solution of $XoR \gtrsim S$. We have

$$\bigvee_v (X(u,v) \wedge R(v,w)) \geqslant S(u,w) \quad \text{for } \forall (u,w).$$

By lemma 1, we have

for $\forall \mathcal{E}_n$, $\forall (u,w)$, $\exists\ v_n^* \in V$ such that

$$X(u,v_n^*) \wedge R(v_n^*,w) \geqslant \bigvee_v (X(u,v) \wedge R(v,w)) \dot{-} \mathcal{E}_n \geqslant S(u,w) \dot{-} \mathcal{E}_n .$$

Thus X is the reachable solution of the inequalities

$XoR \gtrsim S^{(n)}$ for $\forall n$.

This means $X \in \bigcap_{n=1}^{\infty} \{\mathcal{X}^{(n)}\}$.

b) Suppose $X \in \bigcap_{n=1}^{\infty} \{\mathcal{X}^{(n)}\}$. By $X \in \mathcal{X}^{(n)}$ $n=1,2,3,\ldots$

we have for $\forall (u,w), \forall \mathcal{E}_n$, $\exists v^* \in V$ such that

$$X(u,v^*) \wedge R(v^*,w) \geqslant S(u,w) \dot{-} \mathcal{E}_n$$

$$\Longrightarrow \forall (u,v), \forall \mathcal{E}_n, \quad \bigvee_v (X(u,v) \wedge R(v,w)) \geqslant S(u,w) \dot{-} \mathcal{E}_n$$

According to lemma 1, we have

$$\bigvee_v (X(u,v) \wedge R(v,w)) \geqslant \bigvee_{n=1}^{\infty} (S(u,w) \dot{-} \mathcal{E}_n) = S(u,w) \quad \forall (u,w)$$

This means $X \circ R \gtrsim S$. Q.E.D.

D. Solution set of XoR = S

The fuzzy relation equation XoR = S (I) is equivalent to the systems of the inequalities

$$\begin{cases} XoR \subseteq S & \text{(III)} \\ XoR \supseteq S & \text{(II)} \end{cases}$$

By theorem 1, the solution set of $XoR \subseteq S$ is

$$\overline{\mathcal{X}} = \{X \mid X \subseteq \overline{X}\}.$$

By theorem 3. the solution set of $XoR \supseteq S$ is

$$\underline{\mathcal{X}} = \bigcap_{n=1}^{\infty} \bigcup_{g_n \in G_n} \{X \mid X \supseteq Xg_n^{(n)}\}.$$

Hence the solution set of XoR= S is

$$\mathcal{X} = \overline{\mathcal{X}} \cap \underline{\mathcal{X}} = \bigcap_{n=1}^{\infty} \bigcup_{g_n \in G_n} \{X \mid Xg_n^{(n)} \subseteq X \subseteq \overline{X}\}.$$

Lemma 2. For $\forall g_n \in G_n$, $\{X \mid Xg_n^{(n)} \subseteq X \subseteq \overline{X}\} \neq \phi$ if and only if g_n satisfies $g_n \in G_n^* \subseteq G$ where

$$G_n^* = \{g_n \mid U \times W \xrightarrow{g_n} V,\ \text{for } \forall(u,w),\ \overline{X}(u,g\ (u,w)) \wedge R(g\ (u,w),w) \geqslant S(u,w) \dot{-} \varepsilon_n\}$$

Proof. a) Suppose $g_n \in G_n$ and $\{X \mid Xg_n^{(n)} \subseteq X \subseteq \overline{X}\} \neq \phi$.

Then we have

$$Xg_n^{(n)} \subseteq \overline{X} \Longrightarrow \text{for } \forall(u,v),\ \overline{X}(u,v) \geqslant Xg_n^{(n)}(u,v)$$

$$\Longrightarrow \text{for } \forall(u,w),\ \overline{X}(u,g_n(u,w)) \geqslant Xg_n^{(n)}(u,g_n(u,w))$$

$\Longrightarrow$ for $\forall(u,w)$ $\overline{X}(u,g_n(u,w))\wedge R(g_n(u,w),w))$

$$\geqslant Xg_n^{(n)}(u,g_n(u,w))\wedge R(g_n(u,w),w)\geqslant S(u,w) \dot{-} \varepsilon_n .$$

$\Longrightarrow g_n \in G_n^*$

b) Suppose $g_n \in G_n^*$, We have

for $\forall(u,w)$, $\overline{X}(u,g_n(u,w))\wedge R(g_n(u,w),w)\geqslant S(u,w) \dot{-} \varepsilon_n$

$\Longrightarrow \overline{X}(u,g_n(u,w))\geqslant S(u,w) \dot{-} \varepsilon_n$ for $\forall(u,w)$.

If for $\forall(u,v)$, $g_n(u,w')=v$ for some w' we have

$$\overline{X}(u,v)= \overline{X}(u,g_n(u,w')\geqslant S(u,w') \dot{-} \varepsilon_n$$

$$Xg_n^{(n)}(u,v)= \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n)\leqslant \overline{X}(u,v)$$

and if $g_n(u,w)\neq v$ for all $w\in W$, we have

$$Yg_n^{(n)}(u,v)= \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n)=0\leqslant \overline{X}(u,v)$$

thus for $\forall(u,v)$, $Xg_n^{(n)}(u,v)\leqslant\overline{X}(u,v)$. We obtain $Xg_n^{(n)}\subseteq \overline{Y}$. This means $\{X \mid Yg_n^{(n)} \subseteq X \subseteq \overline{X}\}\neq\phi$. Q.E.D.

Lemma 3. The solution set of $XoR= S$ is

$$\mathcal{X}= \bigcap_{n=1}^{\infty} \bigcup_{g_n\in G_n^*} \{X \mid Xg_n^{(n)}\subseteq X \subseteq \overline{X}\}.$$

Finally we obtain

Theorem 4. The solution set of $XoR = S$ is

$$\mathcal{X}= \bigcup_{f\in F}\{X \mid X_f \subseteq X \subseteq \overline{X}\}$$

where

$$F=G_1^* \times G_2^* \times \cdots \times G_n^* \times \cdots$$

$$G_n^*=\{g_n \mid U\times W \xrightarrow{g_n} V, \text{ for } \forall(u,w),\ \overline{X}(u,g_n(u,w))\wedge R(u,w),w) \geqslant S(u,w) \dot{-} \varepsilon_n\}.$$

for $\forall f=(g_1, g_2, \cdots, g_n, \cdots)\in F$, $(g_n\in G_n^*)$

$$X_f(u,v)=\bigvee_n \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n)$$

Proof. a) Suppose that X is any solution of XoR= S. By lamma 3, we have

$$\text{for } \forall n,\ \exists g_n\in G_n^*, \text{ such that } Xg_n^{(n)}\subseteq X\subseteq\overline{X}$$

$$\Longrightarrow \exists f=(g_1, g_2, \cdots, g_n, \cdots)\in F, (g_n\in G_n^*) \text{ such that } X\supseteq Xg_n^{(n)} \text{ for } \forall n.$$

$$\Longrightarrow \text{for } \forall(u,v),\ X(u,v)\geqslant \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n) \text{ for } \forall n$$

$$\Longrightarrow \text{for } \forall(u,v),\ X(u,v)\geqslant \bigvee_n \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n)$$

Let

$$X_f(u,v)=\bigvee_n \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n)$$

thus $X_f\subseteq X\subseteq\overline{X}$. This means

$$X\in\bigcup_{f\in F}\{X \mid X_f\subseteq X\subseteq\overline{X}\}$$

b) Suppose

$$X\in\bigcup_{f\in F}\{X \mid X_f\subseteq X\subseteq\overline{X}\}$$

$\Longrightarrow \exists\, f=(g_1, g_2, \dots, g_n, \dots) \in F\,(g_n \in G_n^*)$ such that $X_f \subseteq X \subseteq \overline{X}$

$\Longrightarrow$ for $\forall (u,v)$, $X(u,v) \geqslant X_f(u,v) = \bigvee_n \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n)$

$\Longrightarrow$ for $\forall (u,v)$, $\forall n$, $X(u,v) \geqslant \bigvee_{g_n(u,w')=v} (s(u,w') \dot{-} \varepsilon_n) = Xg_n^{(n)}(u,v)$

$\Longrightarrow$ for $\forall n$, $\overline{X} \supseteq X \supseteq Xg_n^{(n)}$

$\Longrightarrow X \in \bigcap_{n=1}^{\infty} \bigcup_{g_n \in G_n^*} \{ X \mid Xg_n^{(n)} \subseteq X \subseteq \overline{X} \}$

By lemma 3. X is the solution of $X \circ R = S$. Q.E.D.

X_f is called the quasi— minimum of the fuzzy relation equation $X \circ R = S$ associated with $f \in F$.

E. Example

We consider the fuzzy relation equation

$$(x_1, x_2, \dots, x_n, \dots) \circ \begin{pmatrix} 0.49 \\ 0.499 \\ 0.4999 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = 0.5$$

where

$$X = (x_1, x_2, \dots, x_n, \dots) \in F(\{u\} \times V)$$

$$R = \begin{pmatrix} 0.49 \\ 0.499 \\ 0.4999 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \in F(V \times \{w\})$$

$S = 0.5 \in F(\{u\} \times \{w\})$

$V = \{v_1, v_2, \ldots, v_n, \ldots\}$

By theorem 1, we have

$$\bar{x}_k = \wedge\{\phi\} = 1 \qquad k = 1, 2, \ldots$$

thus $\bar{X} = (1, 1, 1, \ldots)$.

By thecrem 4, we have

$$G^*_n = \left\{ k_n \,\middle|\, \bar{x}_{k_n} \wedge \gamma_{k_n} \geqslant 0.5 - \frac{1}{10^n} \right\}$$

thus

$$G^*_1 = G^*_2 = \{1, 2, 3, \ldots\}, \quad G^*_3 = \{2, 3, 4, \ldots\}, \ \ldots,$$

$$G^*_n = \{n-1, n, n+1, \ldots\}, \ldots$$

$$F = G^*_1 \times G^*_2 \times \ldots \times G^*_n \times \ldots$$

If we select $f = (k_1, k_2, k_3, \ldots) = (1, 2, 3, \ldots) \in F$

thus

$$x_{f_k} = \bigvee_n \bigvee_{k_n = k} \left(0.5 - \frac{1}{10^k}\right) = 0.4\overbrace{9 \ldots 9}^{k} \qquad (k = 1, 2, \ldots)$$

hence

$$X_f = (0.49, 0.499, 0.4999, \ldots)$$

If we select $f = (k_1, k_2, k_3, \ldots) = (2, 2, 3, 4, \ldots) \in F$

thus

$$X_f = (0, 0.499, \quad 0.4999, \ldots)$$

If we select $f = (2, 2, 4, 4, 6, 6, 8, 8, \ldots) \in F$

thus

$X_f = (0, 0.499, 0, 0.49999, 0, 0.4999999, \ldots)$.

There exist infinite quasi—minima, but for any quasi—minimum X_f, there exists another quasi-minimum $X_{f'}$, such that $X_{f'} \subset X_f$

References

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