SOME PROPERTIES OF g_λ-MEASURE

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In this paper, we proved in the first place the quasi—additivity of Sugeno's g_λ-measure. For the g_λ-measure on finite set X, we defined the characteristic function $G_n(\lambda)$ of g_λ-measure, and its some properties are discussed. Finally, we are to show the relation between g_λ-measure and probability measure.

Keywords: Fuzzy measure, Characteristic function, Plausibility measure, Belief function.

1. The quasi-additivity of g_λ-measure

Let X is a non-empty set and $\mathcal{B}$ is a σ-algebra of subsets of X, if set function on $(X,\mathcal{B})$ g: $\mathcal{B} \longrightarrow [0, 1]$ has the following properties:

(i) $g(\phi) = 0$, $g(X) = 1$;

(ii) if A, $B \in \mathcal{B}$, and $A \subset B$, then $g(A) \leq g(B)$;

(iii) if $An \in \mathcal{B}$, and $\{An\}$ is monotone, then $\lim g(An) = g(\lim An)$.

g is called fuzzy measure on $(X,\mathcal{B})$.[1]

If A, $B \in \mathcal{B}$, $A \cap B = \phi$, then

$$g(A \cup B) = g(A) + g(B) + \lambda g(A) \cdot g(B) \tag{1.1}$$

where $\lambda \in (-1, \infty)$, in this time, fuzzy measure g is called g_λ-measure on $(X,\mathcal{B})$, and written g_λ.

We know that for arbitrary a family of disjoint subsets $\{An\}$ in $\mathcal{B}$, g_λ-measure is countably λ-additive fuzzy measure, that is

$$g_\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \begin{cases} \sum_{n=1}^{\infty} g_\lambda(A_n) & \lambda = 0 \\ \frac{1}{\lambda}\left[\prod_{n=1}^{\infty}\left(1+\lambda g_\lambda(A_n)\right) - 1\right] & \lambda \neq 0 \end{cases} \tag{1.2}$$

Definition 1.1. Let X be an arbitrary set, if the sequence of sets $\{An\}$ in X has the following properties:

(i) $\bigcup_{n=1}^{\infty} An = X$;

(ii) for every $i \neq j$, $Ai \cap Aj = \phi$.

$\{An\}$ is called a partition of X.

Theorem 1.1. If $\{An\}$ is a partition of X, g_λ is a g_λ-measure on $(X, \mathcal{B})$, for $\lambda \neq 0$, then

$$\sum_{n=1}^{\infty} \log_{1+\lambda}\left[1+\lambda g_\lambda(A_n)\right] = 1 \tag{1.3}$$

Proof. Since $\{An\}$ is a partition of X, from (1.2), we have

$$\prod_{n=1}^{\infty}\left[1+\lambda g_\lambda(A_n)\right] = 1+\lambda \tag{1.4}$$

We take logarithm for two sides of (1.4), it follows that

$$\sum_{n=1}^{\infty} \log_{1+\lambda}\left[1+\lambda g_\lambda(A_n)\right] = 1$$

(1.3) is called quasi-additivity of g_λ-measure.

Besides, [8] put forward a new proposition: for $\lambda > 0$, g_λ-measure has superadditivity; for $-1 < \lambda < 0$, g_λ-measure has subadditivity; for $\lambda = 0$, g_0 has additivity.

2. The properties of g_λ-measure on finite set

In this section, X denotes finite set.

Definition 2.1. Let $X = \{x_1, x_2, \ldots\ldots x_n\}$, if $g_i = g_i(\{x_i\}) \in [0,1]$, $g_i(\phi) = 0$, $i = 1, 2, \ldots\ldots n$. We say that g_i is fuzzy density on X.

It is easy to show[7], for $\lambda \neq 0$, if fuzzy density gi on X satisfies equation:

$$\frac{1}{\lambda}\left[\prod_{n=1}^{\infty}(1+\lambda g_i)-1\right]=1 \qquad (2.1)$$

then fuzzy density g_i can generate a g_λ-measure, that is, for arbitrary $A\subset X$

$$g_\lambda(A)=\frac{1}{\lambda}\left[\prod_{x_i\in A}(1+\lambda g_i)-1\right] \qquad (2.2)$$

If g_λ is generated by gi (i=1,2,......n), we say that $g_i=g_i(\{x_i\})$ is fuzzy distribution of g_λ, and g_λ is a unique g_λ-measure on $(X,\mathcal{P}(X))$.

Definition 2.2. If g_i is fuzzy distribution of g_λ, we say that

$$G_n(\lambda)=\prod_{i=1}^{n}(1+\lambda g_i)-\lambda-1 \qquad (2.3)$$

is characteristic function of g_λ.

Theorem 2.1. If $g_i>0$ (i=1,2......n) is fuzzy distribution of g_λ, then $Gn(\lambda)$ has not different positive root; and if $Gn(\lambda)$ has positive root $\lambda>0$, then $\sum_{i=1}^{n}g_i<1$.

Proof. By (2.2), we get

$$g_\lambda(\{x_1,x_2,\cdots x_n\})=\frac{1}{\lambda}\left[\prod_{i=1}^{n}(1+\lambda g_i)-1\right]$$
$$=\sum_{i=1}^{n}g_i+\lambda\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}g_{i_1}g_{i_2}+$$
$$\cdots\cdots+\lambda^{n-1}g_1g_2\cdots\cdots g_n .$$

Since $g_\lambda(\{x_1,x_2,\cdots x_n\})=g_\lambda(X)=1$
hence

$$G_n(\lambda)=\lambda\left(\sum_{i=1}^{n}g_i-1\right)+\lambda^2\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}g_{i_1}g_{i_2}+\cdots\cdots+\lambda^n g_1g_2\cdots g_n$$

When $\lambda\neq 0$, $Gn(\lambda)=0$ is equivalent to

$$\left(\sum_{i=1}^{n}g_i-1\right)+\lambda\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n}g_{i_1}g_{i_2}+\cdots\cdots+\lambda^{n-1}g_1g_2\cdots g_n=0 \qquad (2.4)$$

If (2.4) has two different positive roots: $\lambda_1>0$, $\lambda_2>0$, and $\lambda_2<\lambda_1$, then taking λ_1, λ_2 into (2.4) respectively, ferthmore making subtraction, we get

$$(\lambda_1-\lambda_2)\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} g_{i_1} g_{i_2} + \cdots\cdots + (\lambda_1^{n-1}-\lambda_2^{n-1}) g_1 g_2 \cdots g_n = 0 \qquad (2.5)$$

Since $\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} g_{i_1} g_{i_2} > 0, \cdots\cdots g_1 g_2 \cdots g_n > 0$, $\lambda_1^k - \lambda_2^k > 0$ $(k=1 \cdots\cdots n-1)$
the left-hand side of the (2.5) does not equal to zero, this is contradiction from assumed $\lambda_1 > \lambda_2 > 0$. Hence, (2.4) has not different positive roots, therefore, if $G_n(\lambda)$ has positive root, it has only one.

Besides, we assume that (2.4) has a positive root $\lambda > 0$, owing to

$$\sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} g_{i_1} g_{i_2} > 0, \quad \cdots\cdots \quad g_1 g_2 \cdots g_n > 0$$

we have $\sum_{i=1}^{n} g_i < 1$, which completes the proof.

Theorem 2.2. If $g_i > 0$ (i=1,......n) is fuzzy distribution of g_λ, then Gn(λ) has not different negative root; and if Gn(λ) has negative root $\lambda < 0$, then $\sum_{i=1}^{n} g_i > 1$.

Proof. At first, we prove that Gn(λ) has not different negative root.

Since Gn(λ)=o is equivalent to

$$1+\lambda = 1+\lambda\sum_{i=1}^{n} g_i + \lambda^2 \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} g_{i_1} g_{i_2} + \cdots\cdots + \lambda^n g_1 g_2 \cdots g_n \qquad (2.6)$$

we denote the right-hand side of (2.6) by Kn(λ), we first use induction on n to prove $K_n'(\lambda) > 0$, $K_n''(\lambda) > 0$ when $\lambda \in (-1, 0)$.

Obviously, for n=2,

$$K_2(\lambda) = 1 + \lambda(g_1 + g_2) + \lambda^2 g_1 g_2$$

$$K_2'(\lambda) = g_1(1+\lambda g_2) + g_2(1+\lambda g_1) > 0$$

$$K_2''(\lambda) = 2 g_1 g_2 > 0$$

The conclusion is true.

Writing down

$$K_{k+1}(\lambda) = K_k(\lambda) \cdot (1+\lambda g_{k+1})$$

we have

$$K'_{k+1}(\lambda) = K'_k(\lambda)\cdot(1+\lambda g_{k+1}) + g_{k+1}\cdot K_k(\lambda) > 0$$

and

$$K''_{k+1}(\lambda) = K''_k(\lambda)\cdot(1+\lambda g_{k+1}) + 2g_{k+1} K'_k(\lambda) > 0$$

So, if the proposition is valid for k, it is evidently also valid for k+1.

Hence, $K_n(\lambda)$ is monotonically increasing function in the interval $(-1,0)$, So, $K_n(\lambda)$ must cross $\lambda+1$ just once in the interval $(-1,0)$, that is, if $G_n(\lambda)$ has negative root, then it has only one.

We assume that (2.4) has a negative root $\lambda < 0$. Let us define a sequence $\{X_i\}$, $i=1,\ldots,n-1$, of subsets of X:

$$X_i = \{x_{i+1}, x_{i+2}, \cdots\cdots x_n\}$$

Since $\{x_1\} \cup X_1 = X$ and $\{x_1\} \cap X_1 = \Phi$ then in compliance with the definition of g_λ, we have

$$g_1 + g_\lambda(X_1) + \lambda g_1 g_\lambda(X_1) = 1$$

As we assume $\lambda < 0$, which follows that

$$g_1 + g_\lambda(X_1) > 1 \tag{2.7}$$

Further $\{x_2\} \cup X_2 = X_1$ and $\{x_2\} \cap X_2 = \Phi$, so we have

$$g_\lambda(X_1) = g_2 + g_\lambda(X_2) + \lambda g_2 g_\lambda(X_2)$$

As

$$g_2 + g_\lambda(X_2) > g_\lambda(X_1) \tag{2.8}$$

then surely (2.7), (2.8), leads to

$$g_1 + g_2 + g_\lambda(X_2) > 1$$

Recurring and noting that $X_{n-1} = x_n$ we will arrive finally to

$$g_1 + g_2 + \cdots\cdots + g_n > 1$$

The proof of the theorem is complete.

Theorem 2.3: Let $g_i \neq 0 (i=1\ldots n)$ is fuzzy distribution of g_λ, if $\sum_{i=1}^{n} g_i = 1$, then g_λ is probability measure on $(X, \mathcal{P}(X))$.

Proof: Let us define a sequence $\{X_i\}$, $i=1\ldots n-1$ of subsets of the set X:

$$X_i = \{x_{i+1}, x_{i+2}, \ldots x_n\}$$

Since $\{x_1\} \cup X_1 = X$, $\{x_1\} \cap X_1 = \emptyset$, then in compliance with the definition of g_λ. We have

$$g_1 + g_\lambda(X_1) + \lambda g_1 g_\lambda(X_1) = 1 \tag{2.9}$$

Similarly, we have

$$g_\lambda(X_1) = g_2 + g_\lambda(X_2) + \lambda g_2 g_\lambda(X_2) \tag{2.10}$$

We obtain

$$g_1 + g_2 + \lambda g_1 g_2 + (1 + \lambda g_1 + \lambda g_2 + \lambda^2 g_1 g_2) g_\lambda(X_2) = 1$$

from (2.9) and (2.10)

Recurring and noting that $X_{n-1} = x_n$, we will have finally to

$$\sum_{i=1}^{n} g_i + \lambda \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n} g_{i_1} g_{i_2} + \cdots\cdots + \lambda^n g_1 g_2 \cdots g_n = 1$$

Since $\sum_{i=1}^{n} g_i = 1$, therefore

$$\lambda \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n} g_{i_1} g_{i_2} + \cdots\cdots + \lambda^n g_1 g_2 \cdots g_n = 0$$

Because $g_i \neq 0$, it follows that $\lambda = 0$, thus, g_λ is a probability measure on $(X, \mathcal{P}(X))$, hence the conclusion of this theorem holds.

Theorem 2.4: Let $g_i \neq 0 (i=1\ldots n)$ is fuzzy distribution of g_λ,

(i) If $G_n'(0) > 0$, then g_λ is a plausibility measure.

(ii) If $G_n'(0) = 0$ then g_λ is a probability measure.

(iii) If $G_n'(0) < 0$ then g_λ is a belief function.

Proof: (i) Since $G_n'(0) = \sum_{i=1}^{n} g_i - 1 > 0$, by (2.4), $G_n(\lambda)$ only has negative root, i.e. $-1 < \lambda < 0$. Because of [7. Theorem 6.1.5], g_λ is a plausibility measure.

(ii) If $G_n'(0)=\sum_{i=1}^{n} g_i - 1 = 0$, then $\sum_{i=1}^{n} g_i = 1$, according to Theorem 2.3, g_λ is a probability measure.

(iii) If $G_n'(0)=\sum_{i=1}^{n} g_i - 1 < 0$, by (i) and (ii), it follows $\lambda > 0$, because of [7, Theorem 6.1.3], hence g_λ is a belief function.

By Theorem 2.1, Theorem 2.2, Theorem 2.4, we have

Corollary 2.1: g_λ is a plausibility measure if and only if there exists a unique $\lambda \in (-1, 0)$.

g_λ is a belief function if and only if there exists a unique $\lambda \in (0, \infty)$.

Theorem 2.5: Characteristic function $G_n(\lambda)$ has the following properties:

(i) $G_n(0) = 0$

(ii) If $\lambda \in (0, \infty)$, then we have $G_n'(0) < 0$.

(iii) If $\lambda \in (-1, 0)$, then we have $G_n'(0) > 0$.

Proof: The proof is immediate.

3. The Relation Between g_λ-measure and Probability Measure

In [4], Wierzchon proved that a g_λ-measure produces exactly one probability measure on measurable space $(X, \mathcal{B})$, But he said, that the inverse is not true. In this section, we will prove that a probability measure can generates a g_λ-measure.

Theorem 3.1: Let X be a non-empty set and $\mathcal{B}$ be a σ-algebra of subsets of X, if g_λ is a g_λ-measure on $(X, \mathcal{B})$ and $\lambda \neq 0$, then

(i) $g^* = \dfrac{\log(1+\lambda g_\lambda)}{\log(1+\lambda)}$ is a probability measure on $(X, \mathcal{B})$.

(ii) If P is a probability measure on $(X, \mathcal{B})$ and $\lambda \neq 0$, then

$$g_\lambda = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^P$$

is a g_λ-measure on $(X, \mathcal{B})$.

Proof: (i) See [3], [4].

(ii) Since P is a probability measure and $\lambda \neq 0$, then

$$g_\lambda(\Phi) = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(\Phi)} = 0.$$

$$g_\lambda(X) = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(X)} = 1.$$

we assume arbitrary A, $B \in \mathcal{B}$, $A \cap B = \Phi$, then

$$\begin{aligned} g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)\cdot g_\lambda(B) &= -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(A)} - \frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(B)} \\ &\quad + \lambda\left[-\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(A)}\right]\left[-\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(B)}\right\} \\ &= -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(A)+P(B)} \\ &= -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{P(A\cup B)} \\ &= g_\lambda(A\cup B) \end{aligned}$$

Because P is a probability measure and $f(x) = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{x}$ is monotonically nondecreasing, obviously g_λ is continuous. therefore, g_λ is a g_λ-measure on $(X, \mathcal{B})$.

In [1], Sugeno attempt to construct a g_λ-measure on the Borel field $\mathcal{B}$ of R, he use a distribution function of probability measure and define a set function ψ on every half open interval $(a, b] \in \mathcal{B}$

$$\psi((a,b]) = \frac{h(a) - h(b)}{1 + \lambda h(a)} \qquad \text{where } -1 < \lambda < \infty$$

and assert that ψ is a g_λ-measure on $(X, \mathcal{B})$. In [1], it is nothing but to prove that g_λ is a g_λ-measure on semi-ring $\mathcal{P} = \{(a,b]: -\infty < a \leq b < +\infty\}$, yet he can not show that the ψ on $\mathcal{P}$ can be uniqual extened to $(X, \mathcal{B})$. Using Theorem 3.1, we can introduce a g_λ-measure on $(X, \mathcal{B})$, by a distribution function on X=R.

Theorem 3.2: Let X=R, if h(x): $R \to [0,1]$ with the following properties

(i) If $x \leq y$, then $h(x) \leq h(y)$;

(ii) h(x) is right continuous;

(iii) $\lim_{x \to -\infty} h(x) = 0, \lim_{x \to +\infty} h(x) = 1.$

then h(x) can introduce a g_λ-measure on $(X, \mathcal{B})$.

Proof: By [9], h(x) is a distribution function on X, hence, there exist a random variable ξ on a probability space $(X, \mathcal{B}, P)$. his distribution function just is h(x). But, for arbitrary $A \in \mathcal{B}$. $P(\xi \in A)$ can be unique ditermined by the distribution function of ξ, So, h(x) can generates unique probability measure P on $(X, \mathcal{B})$. By using Theorem 3.1 and let

$$g_\lambda = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^P$$

where $\lambda \in (-1, \infty)$, $\lambda \neq 0$. We get a g_λ-measure on $(X, \mathcal{B})$. Thus, the conclusion of this theorem holds.

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