AN EXTENDING THEOREM FOR FUZZY P-MEASURES

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1. Preliminary notions

Let $\sigma \subset \mathbb{F}(\Omega)$ be any soft fuzzy σ-algebra i.e. fuzzy σ-algebra (see [1]) uncontaining the fuzzy subset $[\![\frac{1}{2}]\!]_\Omega : \Omega \to \{\frac{1}{2}\}$, [6]. The fuzzy P-measure on σ is defined as mapping $p: \sigma \to [0,1]$. such that:

- for any $\mu \in \sigma$

$$p(\mu \vee (1-\mu)) = 1 ; \tag{1.1}$$

- if $\{\mu_n\}$ is a finite or an infinite sequence of pairwise W-separated fuzzy subsets (i.e. $\mu_i \leq 1 - \mu_j$ for each pair (i,j) which $i \neq j$ [4]) then

$$p(\sup_n \{\mu_n\}) = \sum_n p(\mu_n) \quad . \; [6] \tag{1.2}$$

Among others things, any fuzzy P-measure p on σ is nondecreasing function fulfilling the following conditions:

$$\forall \mu \in \sigma \qquad \mu \leqslant [\![\tfrac{1}{2}]\!]_\Omega \Rightarrow p(\mu) = 0 , \tag{1.3}$$

$$\forall (\mu,\nu) \in \sigma^2 \qquad p(\mu \vee \nu) + p(\mu \wedge \nu) = p(\mu) + p(\nu) , \tag{1.4}$$

$$\forall_{\substack{(\mu,\nu) \in \sigma^2 \\ \mu \geqslant \nu}} \qquad p(\mu \wedge (1-\nu)) = 0 \Rightarrow p(\mu) = p(\nu) , \tag{1.5}$$

$$\forall \{\mu_n\} \in \sigma^{\mathbb{N}} \qquad \{\mu_n\} \uparrow \mu \in \sigma \Rightarrow \{p(\mu_n)\} \uparrow p \quad . \tag{1.6}$$

Moreover, we have: $p(\mu \wedge \nu) = p(\mu)$ for all $\mu \in \sigma$ iff $p(\nu)$ [6]. The triplet (Ω, σ, p) is called a soft fuzzy probability space.

2. Generalization of ordinary extending theorems.

Let $\hat{\sigma} \subset \mathbb{F}(\Omega)$ be any soft fuzzy algebra i.e. fuzzy algebra uncontaining $\left[\!\left[\frac{1}{2}\right]\!\right]_{\Omega}$. Cover of fuzzy subset μ is defined as the set $C(\mu) = \{\{\mu_n\} \mid \mu \leq \sup_n \{\mu_n\}, \forall\, n \in \mathbb{N}: \mu_n \in \hat{\sigma}\}$ for each $\mu \in \mathbb{F}(\Omega)$. Furthermore, the mapping $p: \hat{\sigma} \to [0,1]$ is a fuzzyP-measure defined on $\hat{\sigma}$, only.

Theorem 2.1: The outer measure p^*, defined by

$$\forall\, \mu \in \mathbb{F}(\Omega) \qquad p^*(\mu) = \inf\{ \sum_n p(\mu_n) : \{\mu_n\} \in C(\mu)\} \quad ,(2.1)$$

is the unique extension of p to the smallest soft fuzzy σ-algebra containing $\hat{\sigma}$, which is a fuzzy P-measure. [3]

Assume now, that $\Omega = \overline{\mathbb{R}} = [-\infty, +\infty]$. Let β_S be infinite Borel family defined in [5]. Each fuzzy subset μ in β_S can be described by the following unions:

$$\mu = \mu_1 = \sup_n \{\varphi\, [a_n, b_n[\,\} \tag{2.2}$$

or

$$\mu = \mu_2 = \sup_n \{\varphi\, [a_n, b_n[\,\} \vee \varphi\, [a_0, +\infty] \tag{2.3}$$

where the mappings $\varphi[a,b[\; : \overline{\mathbb{R}} \to [0,1]$ and $\varphi[a,+\infty] : \overline{\mathbb{R}} \to [0,1]$ are fuzzy intervals defined, for each pair $(a,b) \in \overline{\mathbb{R}}^2$, in [5]. For this case, we have

Theorem 5.2: For each function $F: \overline{\mathbb{R}} \to [0,1]$, fulfilling the conditions:

$$F(-\infty) = 0 \qquad (2.4)\,, \qquad F(+\infty) = 1 \qquad (2.5),$$

$$\forall (x,y) \in \bar{\mathbb{R}}^2 \qquad x \leqslant y \Rightarrow F(x) \leqslant F(y) \qquad (2.6),$$

$$\forall \{x_n\} \uparrow \bar{\mathbb{R}}^{\mathbb{N}} \qquad \{x_n\} \uparrow x \Rightarrow \{F(x_n)\} \uparrow F(x) \qquad (2.7).$$

there exists the unique fuzzy P-measure $p : \beta_\varsigma \to [0,1]$ having the following properties:

$$p\,([+\infty,+\infty]) = 0\,, \qquad (2.8)$$

$$\forall x \in \bar{\mathbb{R}} \qquad p\,([-\infty, x[\,) = F(x)\,. \;[8] \qquad (2.9)$$

Definition 2.1: The projection π_ς on $2^{\mathbb{R}}$ is a mapping

$$\pi_\varsigma : \beta_\varsigma \to 2^{\mathbb{R}}\,,$$

defined by the identity

$$\pi_\varsigma(\mu) = \begin{cases} \bigcup_n [a_n, b_n[\setminus \{-\infty\} & \mu = \mu_1 \\ \bigcup_n [a_n, b_n[\cup [a_0, +\infty[\setminus \{+\infty\} & \mu = \mu_2, \end{cases} \qquad (2.10)$$

where μ_1 and μ_2 are described respectively by (2.2) or (2.3). [7]

Lemma 2.1: The projection π_ς satisfies the following properties:

$$\forall \{\mu_n\} \in \beta_\varsigma^{\mathbb{N}} \qquad \pi_\varsigma(\sup_n \{\mu_n\}) = \bigcup_n \pi_\varsigma(\mu_n)\,, \qquad (2.11)$$

$$\forall (\mu,\nu) \in \beta_\varsigma^2 \qquad \mu \leqslant 1 - \nu \Rightarrow \pi_\varsigma(\mu) \cap \pi_\varsigma(\nu) = \emptyset\,, \qquad (2.12)$$

$$\forall \mu \in \beta_\varsigma \qquad \pi_\varsigma(\mu \vee (1-\mu)) = \mathbb{R}\,. \;[7] \qquad (2.13)$$

Theorem 2.3: Let $F : \bar{\mathbb{R}} \to [0,1]$ be any function fulfilling (2.4), (2.5), (2.6) and (2.7). Then the mapping $p^* : \beta_\varsigma \to [0,1]$, defined by

$$\forall \mu \in \beta_\varsigma \,. \qquad p^*(\mu) = \int_{\pi_\varsigma(\mu)} dF \qquad (2.14)$$

is the unique fuzzy P-measure on β_ς, which satisfies (2.8) and (2.9).

Proof: The conditions (2.1) and (2.2) are immediate consequence of the Lemma 5.1. Also, the property (2.9) is self-evident. Since $\Pi_\xi(\varphi[+\infty,+\infty]) = \emptyset$, the condition (2.8) holds, too. The uniqueness follows from the Theorem 2.2. ■

The last result are more general than analogous thesis presented in [8]. All above theorems are generalization of well-known theorems from ordinary theory of probability spaces, for the fuzzy case.

3. Remarks on fuzzy spaces

Let $\mathfrak{S} \subset \mathbb{F}(\Omega)$ be any soft fuzzy $\mathfrak{S}$-algebra. Since $0_\Omega \in \mathfrak{S}$, the crisp set Ω can be decomposed as union

$$\Omega = \Omega_1 \cup \Omega_2 , \tag{3.1}$$

where $\Omega_1 \neq \emptyset$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\chi_{\Omega_2} \in \mathfrak{S}$ (the mapping χ_{Ω_2} is the mambership function of crisp set Ω_2). Obviously, Ω_2 can be empty. Let Ω_1 be a fixed crisp subset in Ω satisfying (3.1).

Definition 3.1: The mapping

$$K(\cdot,\Omega_1) : \mathbb{F}(\Omega) \to 2^{\Omega_1} ,$$

defined by the identity

$$\forall\ \mu \in \mathbb{F}(\Omega) \quad K(\mu,\Omega_1) = \{\omega : \omega \in \Omega_1,\ \mu(\omega) > \tfrac{1}{2}\}, \tag{3.2}$$

is called a support of nonemptiness in Ω_1. [7]

Definition 3.2: The mapping

$$K^*(\cdot,\Omega_1) : \mathbb{F}(\Omega) \to 2^{\Omega_1} ,$$

given by

$$\forall \mu \in \mathbb{F}(\Omega) \qquad K^*(\mu, \Omega_1) = \{\omega : \omega \in \Omega_1, \mu(\omega) = \tfrac{1}{2}\}, \quad (3.3)$$

is called a support of ill-defined elements in Ω_1. [7]

Let us define the following families of crisp subsets:

$$\mathbb{K}(\Phi, \Omega_1) = \{A : A \in 2^{\Omega_1}, \exists \mu \in \Phi \ : A = K(\mu, \Omega_1) \text{ or } A = L(\mu, \Omega_1)\}, \quad (3.4)$$

$$\mathbb{K}^*(\Phi, \Omega_1) = \{A : A \in 2^{\Omega_1}, \exists \mu \in \Phi \ : A = K^*(\mu, \Omega_1)\} \quad (3.5)$$

for any $\Phi \subset \mathbb{F}(\Omega)$, where the mapping $L(\cdot, \Omega_1) : \mathbb{F}(\Omega) \to 2^{\Omega_1}$ is given by the identity

$$\forall \mu \in \mathbb{F}(\Omega) \qquad L(\mu, \Omega_1) = K(\mu, \Omega_1) \cup K^*(\mu, \Omega_1). \quad (3.6)$$

Theorem 3.1: If σ is a soft fuzzy σ-algebra in Ω then $\mathbb{K}(\sigma, \Omega_1)$ is a crisp σ-algebra in Ω_1. Moreover, then we have $\mathbb{K}^*(\sigma, \Omega_1) \subset \mathbb{K}(\sigma, \Omega_1)$ and

$$\forall \{\mu_n\} \in (\mathbb{F}(\Omega))^{\mathbb{N}} \qquad K(\sup_n \{\mu_n\}, \Omega_1) = \bigcup_n K(\mu_n, \Omega_1), \quad (3.7)$$

$$\forall \{\mu_n\} \in (\mathbb{F}(\Omega))^{\mathbb{N}} \qquad L(\sup_n \{\mu_n\}, \Omega_1) = \bigcup_n L(\mu_n, \Omega_1), \quad (3.8)$$

$$\forall \mu \in \mathbb{F}(\Omega) \qquad K(1-\mu, \Omega_1) = \Omega_1 \setminus L(\mu, \Omega_1), \quad (3.9)$$

$$\forall \mu \in \mathbb{F}(\Omega) \qquad L(1-\mu, \Omega_1) = \Omega_1 \setminus K(\mu, \Omega_1). \ [7] \quad (3.10)$$

On the other side, let us define the family of fuzzy subsets.

$$\mathbb{E}(S, \Omega) = \{\mu : \mu \in \mathbb{F}(\Omega), \exists (A,B) \in S^2, A \subset B, A = K(\mu, \Omega_1) \text{ and } B = L(\mu, \Omega_1)\} \quad (3.11)$$

for any $S \subset 2^{\Omega_1}$. Then we have $\Phi \subset \mathbb{E}(\mathbb{K}(\Phi, \Omega_1), \Omega)$ for each $\Phi \subset \mathbb{F}(\Omega)$ and:

Theorem 3.2: If S is a crisp σ-algebra in Ω_1, then $\mathbb{E}(S, \Omega)$ is a fuzzy σ-algebra in Ω. Furthermore, $[\![\tfrac{1}{2}]\!]_{\Omega} \in \mathbb{E}(S, \Omega)$.

Proof: The identities (3.7), (3.8), (3.9) and (3.10) imply that $\mathbb{E}(S,\Omega)$ is closed under complementation and denumerable union. Also, $K(1_\Omega, \Omega_1) = L(1_\Omega, \Omega_1) = L([\![\frac{1}{2}]\!]_\Omega, \Omega_1) = \Omega_1 \in S$ and $K([\![\frac{1}{2}]\!]_\Omega, \Omega_1) = \emptyset \in S$.■

Moreover, we defined the following subfamily of $\mathbb{E}(\mathbb{K}(\Phi, \Omega_1), \Omega)$

$$\mathbb{E}^*(\Phi, \Omega, \Omega_1) = \{\mu \in \mathbb{E}(\mathbb{K}(\Phi, \Omega_1), \Omega) : \exists\, A \in \mathbb{K}^*(\Phi, \Omega_1), K^*(\mu, \Omega_1) \subset A\} \tag{3.12}$$

for any $\Phi \subset \mathbb{F}(\Omega)$.

Theorem 3.3: If σ is a soft fuzzy σ-algebra in Ω, then $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ is a fuzzy σ-algebra.

Proof: Since $K^*(1-\mu, \Omega_1) = K^*(\mu, \Omega_1)$ for each $\mu \in \mathbb{F}(\Omega)$, $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ is closed under complementation. Let $\{\mu_n\} \subset \subset \mathbb{E}^*(\sigma, \Omega, \Omega_1)$. Then, according with the Theorems 3.1 and 3.2, $\sup_n \{\mu_n\} \in \mathbb{E}(\mathbb{K}(\sigma, \Omega_1), \Omega)$. Furthermore, then we have

$$K^*(\sup_n \{\mu_n\}, \Omega_1) = \bigcap_n K^*(\mu_n, \Omega_1) \subset K^*(\mu_1, \Omega_1) .$$

So, there exists such subset $A \in \mathbb{K}^*(\sigma, \Omega_1)$ that $K(\sup_n \{\mu_n\}, \Omega_1) \subset A$. Thus $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ is closed under denumerable union. Since $K(0_\Omega, \Omega_1) = K^*(0_\Omega, \Omega_1) = \emptyset \in \mathbb{K}^*(\sigma, \Omega_1)$, the family $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ is a fuzzy σ-algebra.■

Last of all, we define family of fuzzy subsets

$$c(\Phi, \Omega_1) = \{\mu, \mu \in \mathbb{F}(\Omega), \exists\, \nu \in \Phi : \mu = \nu \wedge \chi_{\Omega_1}\} \tag{3.13}$$

for any $\Phi \subset \mathbb{F}(\Omega)$. Note that $c(\sigma, \Omega_1) \subset \sigma$ and $c(\sigma, \Omega_1)$ is a fuzzy σ-algebra in Ω_1 .

Theorem 3.4: If σ is such soft fuzzy σ-algebra that $c(\sigma, \Omega_1)$ is a soft fuzzy σ-algebra in Ω_1 then $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ is a soft fuzzy σ-algebra.

Proof: Suppose that $[\![\frac{1}{2}]\!]_{\Omega} \in \mathbb{E}^*(\sigma, \Omega, \Omega_1)$. Therefore, $K^*([\![\frac{1}{2}]\!]_{\Omega}, \Omega_1) = \Omega_1 \in \mathbb{K}^*(\sigma, \Omega_1)$. So, $[\![\frac{1}{2}]\!]_{\Omega} \wedge \chi_{\Omega_1} \in$ $\in c(\sigma, \Omega_1)$. ■

Futhermore, we observe that

$$\mathbb{K}(\sigma, \Omega_1) = \mathbb{K}(c(\sigma, \Omega_1), \Omega_1) \ , \tag{3.14}$$

$$\mathbb{K}^*(\sigma, \Omega_1) = \mathbb{K}^*(c(\sigma, \Omega_1), \Omega_1) \ . \tag{3.15}$$

4. Fuzzy extension

Let (Ω, σ, p) be such soft fuzzy probability spaces that:

- there exists such crisp subset Ω_1 that the set Ω can be decomposed as union $\Omega = \Omega_1 \cup \Omega_2$ according with (3.1) and $c(\sigma, \Omega_1)$ is a soft fuzzy σ-algebra in Ω_1 ;
- the fuzzy P-measure p on σ satisfies

$$p(\chi_{\Omega_2}) = 0 \ . \tag{4.1}$$

Lemma 4.1: The mapping $p_c: c(\sigma, \Omega_1) \to [0,1]$, given by

$$\forall \mu \in \sigma \qquad p_c(\mu \wedge \chi_{\Omega_2}) = p(\mu) \tag{4.2}$$

is explicitly defined fuzzy P-measure on $c(\sigma, \Omega_1)$.

Proof: Let $\nu \in c(\sigma, \Omega_1)$. Assume that there exists $(\mu_1, \mu_2) \in \sigma^2$ such that $\mu_1 \neq \mu_2$ and $\nu = \mu_1 \wedge \chi_{\Omega_1} = \mu_2 \wedge \chi_{\Omega_1}$. Using (1.2), we get

$p_c(\nu) = p(\mu_1) = p(\mu_1 \wedge (\chi_{\Omega_1} \vee \chi_{\Omega_2})) = p(\mu_1 \wedge \chi_{\Omega_1}) + p(\mu_1 \wedge \chi_{\Omega_2}) = p(\mu_1 \wedge \chi_{\Omega_1}) = p(\mu_2 \wedge \chi_{\Omega_1}) = p(\mu_2 \wedge \chi_{\Omega_1}) + p(\mu_2 \wedge \chi_{\Omega_2}) = p(\mu_2)$

So, the mapping p_c is given explicitly. The conditions (1.1) and (1.2) are self-evident. ■

Lemma 4.2: If pair $(\mu,\nu)\in\mathfrak{S}^2$ satisfies

$$K(\mu,\Omega_1)=L(\nu,\Omega_1)\quad, \tag{4.3}$$

or

$$L(\mu,\Omega_1)=L(\nu,\Omega_1)\quad, \tag{4.4}$$

or

$$K(\mu,\Omega_1)=K(\nu,\Omega_1) \tag{4.5}$$

then $p(\mu)=p(\nu)$.

Proof: If the pair (μ,ν) satisfies (4.3) or (4.4) then we have $\nu(\omega)\geqslant\mu(\omega)\wedge\nu(\omega)\geqslant\frac{1}{2}$ for each $\omega\in\Omega_1$ such that $\mu(\omega)\vee\nu(\omega)\geqslant\frac{1}{2}$. Moreover, the condition (4.5) implies that $\nu(\omega)>\frac{1}{2}$ and $\mu(\omega)\wedge\nu(\omega)>\frac{1}{2}$ for such $\omega\in\Omega_1$ that $\mu(\omega)\vee(\omega)>\frac{1}{2}$. So, for each pair $(\mu,\nu)\in\mathfrak{S}^2$ fulfilling (4.3) or (4.4) or (4.5), we get:

$$((\mu\wedge\chi_{\Omega_1})\vee(\nu\wedge\chi_{\Omega_1}))\wedge(1-\nu\wedge\chi_{\Omega_1})\leqslant[\![\tfrac{1}{2}]\!]_{\Omega}\,.$$

$$(\nu\wedge\chi_{\Omega_1})\wedge(1-(\mu\wedge\chi_{\Omega_1})\wedge(\nu\wedge\chi_{\Omega_1}))\leqslant[\![\tfrac{1}{2}]\!]_{\Omega}\,.$$

This, along with (1.3), (1.4) and (1.5), gives

$$\begin{aligned}p_c(\nu\wedge\chi_{\Omega_1}) &= p_c((\mu\wedge\chi_{\Omega_1})\vee(\nu\wedge\chi_{\Omega_1})) = \\ &= p_c(\mu\wedge\chi_{\Omega_1})+p_c(\nu\wedge\chi_{\Omega_1})-p_c((\mu\wedge\chi_{\Omega_1})\wedge(\nu\wedge\chi_{\Omega_1})) \\ p_c &= p_c\end{aligned}$$

.

The result, together with the Lemma 4.1, shows $p(\mu)=p_c(\mu\wedge\chi_{\Omega_1})=$ $=p_c(\nu\wedge\chi_{\Omega_1})=p(\nu)$. ■

Lemma 4.3: If pair $(\mu,\nu)\in\mathfrak{S}^2$ fulfils

$$K(\mu,\Omega_1)\subset K(\nu,\Omega_1) \tag{4.6}$$

or

$$L(\mu,\Omega_1)\subset L(\nu,\Omega_1) \tag{4.7}$$

or

$$L(\mu,\Omega_1)\subset K(\nu,\Omega_1) \tag{4.8}$$

or

$$K(\mu,\Omega_1)\subset L(\nu,\Omega_1) \tag{4.9}$$

then $p(\mu)\leqslant p(\nu)$.

Proof: For any pair $(\mu,\nu)\in\sigma^2$ we have:

- if (4.6) or (4.8) then $K(\mu\vee\nu,\Omega_1)=K(\nu,\Omega_1)$,
- if (4.7) then $L(\mu\vee\nu,\Omega_1)=L(\nu,\Omega_1)$.

So, for these cases, in agreement with the Lemma 4.2, we get $p(\nu)=p(\mu\vee\nu)\geqslant p(\mu)$. Furthermore, the condition (4.9) implies $\mu(\omega)\wedge\nu(\omega)\geqslant\frac{1}{2}$ for each $\omega\in\Omega_1$ such that $\mu(\omega)>\frac{1}{2}$. Thus

$$(\mu\wedge\chi_{\Omega_1})\wedge(1-(\mu\wedge\chi_{\Omega_1})\wedge(\nu\wedge\chi_{\Omega_1}))\leqslant[\![\tfrac{1}{2}]\!]_\Omega .$$

This, along with (1.3), (1.5) and (4.2) shows that $p(\mu)=$ $=p_c(\mu\wedge\chi_{\Omega_1})=p_c((\mu\wedge\chi_{\Omega_1})\wedge(\nu\wedge\chi_{\Omega_1}))\leqslant p_c(\nu\wedge\chi_\Omega)=p(\nu)$. ∎

Theorem 4.1: The mapping $P:\mathbb{K}(\sigma,\Omega_1)\to[0,1]$ defined by

$$\forall A\in\mathbb{K}(\sigma,\Omega_1)\qquad P(A)=\begin{cases}p(\nu) & A=K(\nu,\Omega_1)\\ p(\mu) & A=L(\mu,\Omega_1)\,,\end{cases}\tag{4.10}$$

is usual probability measure on $\mathbb{K}(\sigma,\Omega_1)$ satisfying the condition

$$\forall A\in\mathbb{K}^*(\sigma,\Omega_1)\qquad P(A)=0\,.\tag{4.11}$$

Proof: The Lemma 4.2 shows that the mapping P is explicitly defined by (4.10). Since $\Omega_1=L(\mu\vee(1-\mu)),\Omega_1)$ for any $\mu\in\sigma$, by (1.1) we get $P(\Omega_1)=1$.

Let $\{A_n\}$ be sequence of pairwise disjoint subsets in $\mathbb{K}(\sigma,\Omega_1)$. Then there exists such sequence $\{\mu_n\}\in\sigma^{\mathbb{N}}$ that $A_n=K(\mu_n,\Omega_1)$ or $A_n=L(\mu_n,\Omega_1)$ for each positive integer n. Note that the fuzzy subsets $\{\mu_n\}$ are mutually W-separated. The Lemma 4.3 implies that the mapping P is nondecreasing. Therefore, by (3.7), (3.8) and (1.2), we obtain

$$\sum_n P(A_n)=\sum_n p(\mu_n)=p(\sup_n\{\mu_n\})=P(K(\sup_n\{\mu_n\},\Omega_1))=$$

$$=P(\bigcup_n K(\mu_n,\Omega_1))\leqslant P(\bigcup_n A_n)\leqslant P(\bigcup_n L(\mu_n,\Omega_1))=$$

$= P(L(\sup_n \{\mu_n\}, \Omega_1)) = p(\sup_n \{\mu_n\}) = \sum_n p(\mu_n) = \sum_n P(A_n)$.

So, P is an usual probability measure on $\mathbb{K}(\sigma, \Omega_1)$. Also the condition (4.11) holds because

$P(K^*(\mu, \Omega_1)) = P(L(\mu, \Omega_1) \setminus K(\mu, \Omega_1)) = P(L(\mu, \Omega_1)) - P(K(\mu, \Omega_1)) =$
$= p(\mu) - p(\mu) = 0$

for all $\mu \in \sigma$. ■

Theorem 4.2: Let $P: \mathbb{K}(\sigma, \Omega_1) \to [0,1]$ be an usual probability measure on $\mathbb{K}(\sigma, \Omega_1)$ fulfilling (4.11). Then the mapping $\bar{p}: \sigma \to [0,1]$, defined by means of the identity

$$\bar{p}(\mu) = P(K(\mu, \Omega_1)) \tag{4.12}$$

for all $\mu \in \sigma$, is a fuzzy P-measure on σ which satisfies (4.1). [7]

Theorem 4.3: The mapping $\bar{p}: \mathbb{E}^*(\sigma, \Omega, \Omega_1) \to [0,1]$, defined by (4.10) and (4.12) for each $\mu \in \mathbb{E}^*(\sigma, \Omega, \Omega_1)$, is the unique extension of fuzzy P-measure p on σ to $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$, which is a fuzzy P-measure on $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$.

Proof: Since the mapping P is nondecreasing, the condition (4.11) holds for all $A \in \mathbb{K}^*(\mathbb{E}^*(\sigma, \Omega, \Omega_1), \Omega_1)$. So, according to the Theorem 4.2 the mapping $\bar{p}$ is a fuzzy P-measure on $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ because $\mathbb{K}(\mathbb{E}^*(\sigma, \Omega, \Omega_1), \Omega_1) = \mathbb{K}(\sigma, \Omega_1)$.

Moreover, we have

$\bar{p}(\mu) = P(K(\mu, \Omega_1)) = p(\mu)$

for each $\mu \in \sigma$.

Let $\tilde{p}: \mathbb{E}^*(\sigma, \Omega, \Omega_1) \to [0,1]$ be any fuzzy P-measure on $\mathbb{E}^*(\sigma, \Omega, \Omega_1)$ fulfilling $\tilde{p}(\mu) = p(\mu)$ for each $\mu \in \sigma$. Then we get $\tilde{p}(\mu) = \tilde{P}(K(\mu, \Omega_1))$, where the mapping $\tilde{P}: \mathbb{K}(\mathbb{E}^*(\sigma, \Omega, \Omega_1), \Omega_1) \to [0,1]$ is given by

$$\widetilde{P}(A) = \begin{cases} \widetilde{p}(\mu) & A = K(\mu,\Omega_1) \\ \widetilde{p}(\nu) & A = L(\nu,\Omega_1) \end{cases}$$

for each $A \in \mathbb{K}(\mathbb{E}^*(\sigma,\Omega,\Omega_1),\Omega_1)$. Also we have $\widetilde{P}(A) = 0$ for each $A \in \mathbb{K}^*(\mathbb{E}^*(\sigma,\Omega,\Omega_1),\Omega_1)$. If $\mu \in \mathbb{E}^*(\sigma,\Omega,\Omega_1)$, then there exists such fuzzy subset $\nu \in \sigma$ that $K(\mu,\Omega_1) = K(\nu,\Omega_1)$ or $K(\mu,\Omega_1) = L(\nu,\Omega_1)$. Therefore, we get

$\widetilde{p}(\mu) = \widetilde{P}(K(\mu,\Omega_1)) = \widetilde{P}(K(\nu,\Omega_1)) = \widetilde{p}(\nu) = p(\nu) = P(K(\nu,\Omega_1)) = P(K(\mu,\Omega_1)) = \overline{p}(\mu)$

or

$\widetilde{p}(\mu) = \widetilde{P}(K(\mu,\Omega_1)) = \widetilde{P}(L(\nu,\Omega_1)) = \widetilde{p}(\nu) = p(\nu) = P(L(\nu,\Omega_1)) = P(K(\mu,\Omega_1)) = \overline{p}(\mu)$. ∎

The Theorem 3.2 says that each fuzzy P-measure on σ cannot be extended to $\mathbb{E}(\mathbb{K}(\sigma,\Omega_1),\Omega)$. Nevertheless, for this case we have:

Theorem4.4: The mapping $\overline{p}: \mathbb{E}(\mathbb{K}(\sigma,\Omega_1),\Omega) \to [0,1]$, defined by (4.10) and (4.12) for each $\mu \in \mathbb{E}(\mathbb{K}(\sigma,\Omega_1),\Omega)$ is a fuzzy probability measure (in sense given by Klement at.el. [2]) on $\mathbb{E}(\mathbb{K}(\sigma,\Omega_1),\Omega)$ i.e. the mapping $\overline{p}$ has the following properties:

$\overline{p}(0_\Omega) = 0$ (4.13) ; $p(1_\Omega) = 1$ (4.14)

and (1.4), (1.6) for all fuzzy subsets in $\mathbb{E}(\mathbb{K}(\sigma,\Omega_1),\Omega)$.

Proof: Since $\{0_\Omega, 1_\Omega\} \subset \mathbb{E}^*(\sigma,\Omega,\Omega_1)$, the conditions (4.13) and (4.14) follow from (1.1) and (1.3). Let $(\mu,\nu) \in (\mathbb{E}(\mathbb{K}(\sigma,\Omega_1),\Omega))^2$. Then, by (4.12) we get

$\overline{p}(\mu \vee \nu) = P(K(\mu \vee \nu,\Omega_1)) = P(K(\mu,\Omega_1) \cup K(\nu,\Omega_1)) =$
$= P(K(\mu,\Omega_1)) + P(K(\nu,\Omega_1)) - P(K(\mu,\Omega_1) \cap K(\nu,\Omega_1)) =$
$= \overline{p}(\mu) + \overline{p}(\nu) - P(K(\mu \wedge \nu,\Omega_1)) = \overline{p}(\mu) + \overline{p}(\nu) - \overline{p}(\mu \wedge \nu)$.

So, the condition (1.4) holds. Moreover, if $\{\mu_n\}$ is such non-decreasing sequence of fuzzy subsets in $\mathbb{E}(\mathbb{K}(\mathfrak{S},\Omega_1),\Omega)$ that $\{\mu_n\}\uparrow \mu \in \mathbb{E}(\mathbb{K}(\mathfrak{S},\Omega_1),\Omega)$, then $\{K(\mu_n,\Omega_1)\}\uparrow$ $\uparrow K(\mu,\Omega_1)$. Thus

$\{\bar{p}(\mu_n)\} = \{P(K(\mu_n,\Omega_1))\}\uparrow P(K(\mu,\Omega_1)) = \bar{p}(\mu)$. ■

Theorem 4.5: Let $P: \mathbb{K}(\mathfrak{S},\Omega_1)\to [0,1]$ be an usual probability measure on $\mathbb{K}(\mathfrak{S},\Omega_1)$ fulfilling (4.11). Then the mapping $\hat{p}$: $\mathfrak{S}\to [0,1]$, defined by the identity

$$\hat{p}(\mu) = P(L(\mu,\Omega_1)) \tag{4.15}$$

for all $\mu\in\mathfrak{S}$, is a fuzzy P-measure on $\mathfrak{S}$ which satisfies (4.1) and

$$\forall\mu\in\mathfrak{S} \qquad \hat{p}(\mu) = \bar{p}(\mu) \quad . \quad [7] \tag{4.16}$$

Theorem 4.6: The mapping $\hat{p}: \mathbb{E}^*(\mathfrak{S},\Omega,\Omega_1)\to [0,1]$, defined by (4.10) and (4.15) for each $\mu\in\mathbb{E}^*(\mathfrak{S},\Omega,\Omega_1)$, is the unique extension of fuzzy P-measure p on $\mathfrak{S}$ to $\mathbb{E}^*(\mathfrak{S},\Omega,\Omega_1)$ which is a fuzzy P-measure on $\mathbb{E}^*(\mathfrak{S},\Omega,\Omega_1)$.

Proof: By analogous way, as the proof of the Theorem 4.3, we show that $\hat{p}$ is a fuzzy P-measure on $\mathbb{E}^*(\mathfrak{S},\Omega,\Omega_1)$ which is a extension of p. The uniqueness follows from the Theorem 4.3. ■

Theorem 4.7: The mapping $\hat{p}: \mathbb{E}(\mathbb{K}(\mathfrak{S},\Omega_1),\Omega)\to [0,1]$, defined by (4.10) and (4.15) for each $\mu\in\mathbb{E}(\mathbb{K}(\mathfrak{S},\Omega_1),\Omega)$, is a fuzzy probability measure on $\mathbb{E}(\mathbb{K}(\mathfrak{S},\Omega_1),\Omega)$.

Proof: By analogous way, as the Theorem 4.4. ■

Remark: Comparise the mappings $\bar{p}$ and $\hat{p}$. Since $\bar{p}([\![\tfrac{1}{2}]\!]_\Omega) =$ $= P(K([\![\tfrac{1}{2}]\!]_\Omega,\Omega_1)) = P(\emptyset) = 0 < 1 = P(\Omega_1) = P(L([\![\tfrac{1}{2}]\!]_\Omega,\Omega_1)) =$ $= \hat{p}([\![\tfrac{1}{2}]\!]_\Omega)$, $\bar{p}$ and $\hat{p}$ are different fuzzy probability measure on $\mathbb{E}(\mathbb{K}(\mathfrak{S},\Omega_1),\Omega)$. Moreover, the monotonicity of P

implies that $\overline{p}(\mu) \leq \hat{p}(\mu)$ for all $\mu \in \mathbb{E}(\mathbb{K}(\sigma, \Omega_1), \Omega)$. Therefore, the mappings $\overline{p}$ and $\hat{p}$ are called respectively lover extension of p and higher extension of p . Since $\sigma = \mathbb{E}^*(\sigma, \Omega, \Omega) = \mathbb{E}(\mathbb{K}(\sigma, \Omega), \Omega)$ for the crisp case, presented above results are commonplace for this case. In fuzzy case, the lover and higher extension are necessary for investigation of distributions of fuzzy random variables.

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