Properties of fuzzy implication operators

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1. Abstract.

In this paper we discuss both forward implication and backward implication, and the difference between them is defined. We introduce some properties of fuzzy implication operators, and show the expectation, the variance and the distribution of each fuzzy implication operator, assuming that the two propositions in a given compound proposition are independent of each other and the truth values of the propositions are uniformly distributed on the interval [0,1].

2. Analytical view of fuzzy implication operators

In classical two-valued logic, one wishes a truth-functional connection, which evaluates the logical formulas of two or more propositions (e. g. "p and q", "p or q" and "if p then q") and their truth values are either true or false. Multiple-valued logic is required in the theory of fuzzy sets and relations. One wishes to manipulate the degrees of truth which attach to fuzzy statements. The following discussion is related to implication and introduces some properties of the fuzzy implication operators listed by Bandler and Kohout [1] (see definition 2.2).

Before we discuss multiple-valued implication, let us look at the standard Boolean operators on the set B = {0,1}.

Definition 2.1.

Let p and q be propositions, and v(p) and v(q) be the truth values of p and q, respectively

1. conjunction : v(p and q) = min(v(p),v(q))
2. disjunction : v(p or q) = max(v(p),v(q))
3. negation : v(not p) = 1 - v(p)
4. implication : v(p -> q), given by

v(p) / v(q)	0	1
0	1	1
1	0	1

Since Zadeh introduced fuzzy sets and suggested using min(v(p),v(q)), max(v(p),v(q)) and 1 - v(p) for conjunction, disjunction and negation, respectively, in the fuzzy situation, many authors have proposed other possibilites for these operators [2].

Ten fuzzy implication operators are defined in Def. 2.2. All such operators have truth values in the closed real interval [0,1]. A fuzzy implication operator, ->, is a binary operation from [0,1] x [0,1] into [0,1], which is a generalization of Boolean implication, that is, the values assigned in the crisp "corners", where the values v(p) of p and v(q) of q are zero (false) or one (true), must accord with those of classical Boolean logic.

Definition 2.2.

Let $a = v(p)$ and $b = v(q)$, where p and q are propositions. Let $r = v(p \rightarrow q)$.

1. standard sharp
   $r = 1$ if $a < 1$ or $b = 1$
   $\quad\;\, 0$ otherwise
2. standard strict
   $r = 1$ if $a \le b$
   $\quad\;\, 0$ otherwise
3. standard star
   $r = 1$ if $a \le b$
   $\quad\;\, b$ otherwise
4. Gaines 43
   $r = \min(b/a, 1)$, where $0/0 = 1$

4.5 Modified Gaines 43
   $r = \min(1, b/a, (1-a)/(1-b))$, where if $b = 1$ then $r = 1$

5. Lukasiewicz
   $r = \min(1 - a + b, 1)$

5.5 Kleene-Dienes-Lukasiewicz
   $r = 1 - a + a\,b$

6. Kleene-Dienes
   $r = \max(b, 1 - a)$
7. Early Zadeh
   $r = \max(\min(a, b), 1 - a)$
8. Willmott
   $r = \min(\max(1 - a, b), \max(a, 1 - b, \min(b, 1 - a)))$

The operators are listed in order of increasing fuzziness, and fall into three groups.
Operators 1 and 2 are crisp-valued, operators 3 through 5, while truly fuzzy, have more than half of their values (for $a \le b$) equal to one. Finally operators 5.5 through yield values of one only for $a = 0$ or $b = 1$.

Operator 1 is too severe to find much favor, while operator 4 was introduced by Goguen [4], but Gaines [3] noticed that this implication bears a formal resemblance to conditional probability since, using Zadeh's definition for v(p and q), $v(p \rightarrow q) = v(p \text{ and } q) / v(p)$, while conditional probability[5] is given by $P(q \text{ given } p) = P(p \text{ and } q) / P(p)$.

Sometimes, we may suspect that two propositions p and q are related, but we do not know a priori whether it makes more sense to consider $p \rightarrow q$ or $q \rightarrow p$. Therefore, we would compare to truth of $p \rightarrow q$ to that of $q \rightarrow p$. Let

$$d = v(p \rightarrow q) - v(q \rightarrow p),$$

so $d > 0$ if $p \rightarrow q$ is truer than $q \rightarrow p$. We next present formulas for d for each of the ten operators in Def. 2.2. In order to establish these formulas, we use the following lemma.

Lemma 2.3.

Let x, y and z be real numbers. Then the following equalities hold.

1. $x - \min(y, z) = \max(x - y, x - z)$
2. $x - \max(y, z) = \min(x - y, x - z)$
3. $\min(x, y) - z = \min(x - z, y - z)$
4. $\max(x, y) - z = \max(x - z, y - z)$

Theorem 2.4.

Let $a, b = [0,1]$ and let $p \rightarrow q$, $p \leftarrow q$ be the fuzzy implication from p to q and from q to p, respectively. Let $r1 = v(p \rightarrow q)$ and $r2 = v(p \leftarrow q)$, let $d = r1 - r2$.

Then, for all three implication operators 5, 5.5 and 6 of definition 2.2, $d = b - a$, where $a = v(p)$ and $b = v(q)$.

proof.

For operator 5,

$d = \min(1,\ 1 - a + b) - \min(1,\ 1 - b + a)$

$= \max(\min(1,\ 1 - a + b) - 1,\ \min(1, 1 - a + b) - (1 - b + a))$

$= \max(\min(0,\ b - a),\ \min(b - a,\ 2(b - a)))$

case 1. $(b > a)$ $d = \max(0,\ b - a) = b - a$

case 2. $(b \leq a)$ $d = \max(b - a, 2(b - a)) = b - a$

Therefore, $d = b - a$

For operator 5.5,

$d = (1 - a + a b) - (1 - b + a b) = b - a$

For operator 6,

$d = \max(1 - a,\ b) - \max(1 - b,\ a)$

$= \max(1 - a - \max(1 - b,\ a),\ b - \max(1 - b,\ a))$

$= \max(\min(b - a,\ 1 - 2a),\ \min(2b - 1,\ b - a))$

case 1. $(b \leq 1 - a) \Rightarrow$

$(b - a \leq 1 - 2a)$ and $(b - a \geq 2b - 1)$

$d = \max(1 - 2a,\ b - a) = b - a$

case 2. $(b > 1 - a) \Rightarrow$

$(b - a > 1 - 2a)$ and $(b - a < 2b - 1)$

$d = \max(1 - 2a,\ b - a) = b - a$

Therefore, $d = b - a$

Theorem 2.5.

Let $r1 = v(p \rightarrow q)$ and $r2 = v(p \leftarrow q)$. Let $a = v(p)$ and $b = v(q)$. Let $d = r1 - r2$. Then, for all fuzzy implication operators of definition 2.2, the following relations hold.

1. $r1 - r2 < 0 \Rightarrow b - a < 0$
2. $r1 - r2 > 0 \Rightarrow b - a > 0$
3. $b - a > 0 \Rightarrow r1 - r2 \geq 0$
4. $b - a < 0 \Rightarrow r1 - r2 \leq 0$

proof.

For operator 1,

i)	$0 \leq a < 1,$	$0 \leq b < 1$	:	$d = 0$
ii)	$a = 1,$	$b < 1$	:	$d = -1$
iii)	$a < 1,$	$b = 1$	:	$d = 1$
iv)	$a = 1,$	$b = 1$	:	$d = 0$

For operator 2,

i)	$a < b$	:	$d = 1$
ii)	$a = b$	:	$d = 0$
iii)	$a > b$	:	$d = -1$

For operator 3,

i)	$a < b$	:	$d = 1 - a > 0$
ii)	$a = b$	:	$d = 0$
iii)	$a > b$	:	$d = b - 1 < 0$

For operator 4,

i)	$a < b$	:	$d = 1 - a / b > 0$
ii)	$a = b$	:	$d = 0$
iii)	$a > b$	:	$d = b / a - 1 < 0$

For operator 4.5,

i) $a < b$: $r1 = 1,\ r2 = \min(a/b, (1-b)/(1-a)) < 1$

Therefore, $d = r1 - r2 > 0$

ii) $a = b$: $d = 0$

iii) $a > b$: $r1 = \min(b/a, (1-a)/(1-b)) < 1,\ r2 = 1$

Therefore, $d = r1 - r2 < 0$

For operator 5, 5.5 and 6, see theorem 2.4.

For operator 7,

i) $a < b,\quad a < 1/2$: $d \geq \min(1-2a, b-a) > 0$

ii) $a < b,\quad a \geq 1/2$: $d = 0$

iii) $a = b$: $d = 0$

iv) $a > b,\ 1 > b \geq 1/2$: $d = 0$

v) $a > b,\quad b \leq 1/2$: $d = \max(2b-1, b-a) < 0$

For operator 8,

i) $b \leq 1-a,\ a < b,\quad b > 1/2$: $d = 2b - 1 > 0$

ii) $b \leq 1-a,\ a < b,\ a \leq 1/2,\ b \leq 1/2$: $d = 0$

iii) $b \leq 1-a,\ a = b$: $d = 0$

iv) $b \leq 1-a,\ a > b,\ a \leq 1/2,\ b < 1/2$: $d = 0$

v) $b \leq 1-a,\ a > b,\ a \geq 1/2,\ b \geq 1/2$: $d = b - a < 0$

vi) $b \leq 1-a,\ a > b,\ a > 1/2,\ b \leq 1/2$: $d = 1 - 2a < 0$

vii) $b \geq 1-a,\ a < b,\ a < 1/2,\ b \geq 1/2$: $d = 1 - 2a > 0$

viii) $b > 1-a,\ a < b,\ a < 1/2,\ b \leq 1/2$: $d = b - a > 0$

ix) $b > 1-a,\ a < b,\ a \geq 1/2,\ b > 1/2$: $d = 0$

x) $b > 1-a,\ a = b$: $d = 0$

xi) $b > 1-a,\ a > b,\ a < 1/2,\ b < 1/2$: $d = b - a < 0$

xii) $b > 1-a,\ a > b,\ a \geq 1/2,\ b < 1/2$: $d = 2b - 1 < 0$

xiii) $b > 1-a,\ a > b,\ a \geq 1/2,\ b \geq 1/2$: $d = 0$

Theorem 2.4 and 2.5 can be illustrared by the difference diagrams of the fuzzy implication operators (see fig. 2.6).

For each operator of fig. 2.6, we show a graph of v(p -> q) for various combinations of v(p) and v(q). The abcissa and ordinate are the v(p) and v(q) axes on the closed interval [0,1], respectively. Some of the operators change their functional form across the lines $b = a$, $b = 1 - a$, $a = 1/2$ and/or $b = 1/2$. The difference, $d =$ v(p -> q) - v(p <- q), compares the two directions of implication. If $d > 0$, then p -> q is truer than q -> p.

Based on the behavior of the graphs in fig. 2.6, we summarize the following observations in table 2.7.

table 2.7

	statement	applicability to operators
1.	$d = 0$ if $a = b$	all
2.	$d = 0$ iff $a = b$	2 - 6
3.1	$d \geq 0$ if $a < b$	all
3.2	$d \leq 0$ if $a > b$	all
4.1	$d > 0$ iff $a < b$	2 - 6
4.2	$d < 0$ iff $a > b$	2 - 6
5.	v(p -> q) and d are discontinuous approaching the line $a = b$	2,3
6.	v(p -> q) and d are discontinuous at one or more corner points.	1 - 4.5
7.	$d = b - a$	5,5.5,6
8.	v(p -> q) and d are everywhere continuous and obey statements 1 - 4	5,5.5,6

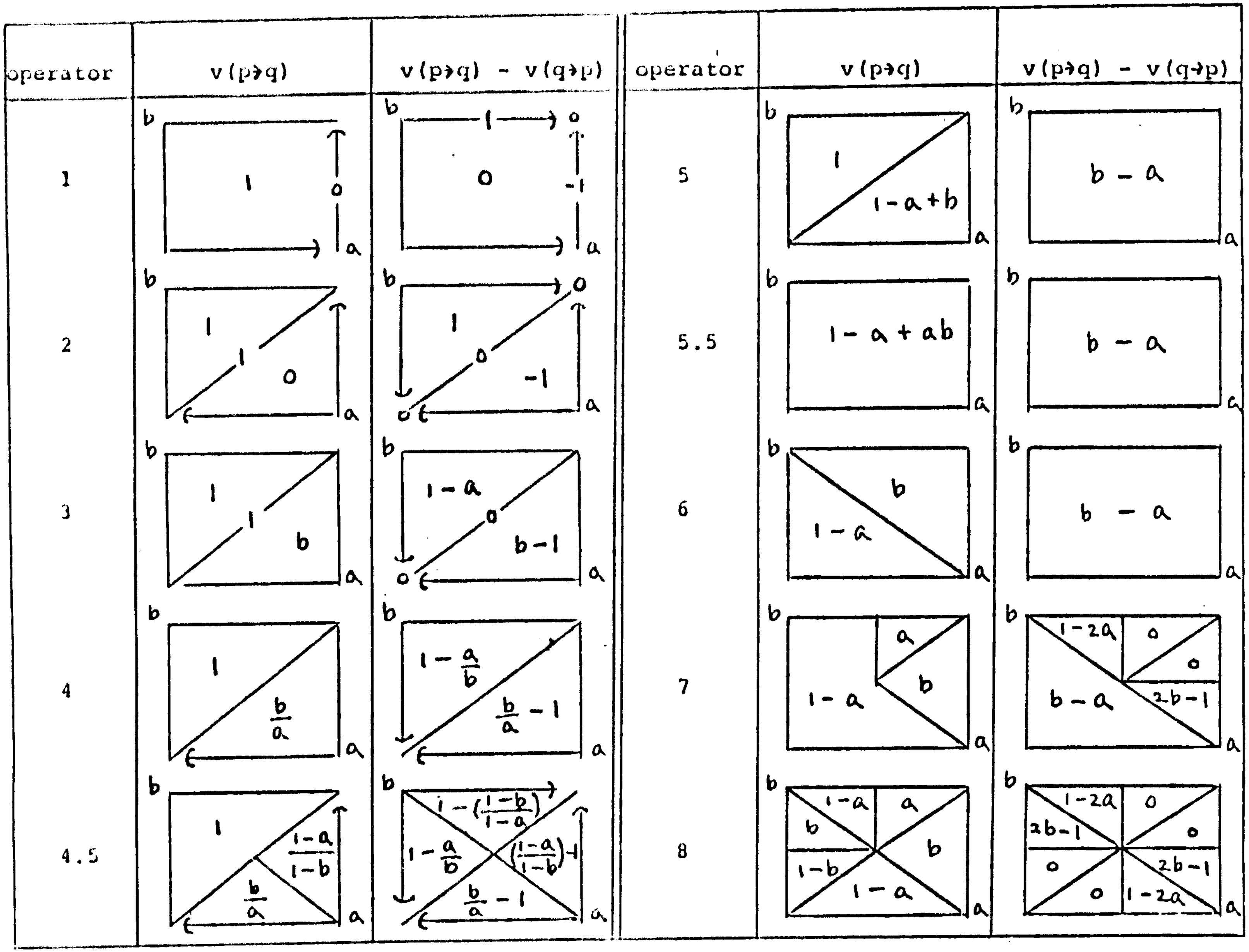


Figure 2.6

→ means the open set.

The observations in table 2.7 lead to the following conclusions.

a. When operators 2 - 6 are used, p -> q is truer than q -> p iff q is truer than p, based on the statement 4.

b. Operators 5, 5.5 and 6 are the best behaved, based on the statements 7 and 8.

3. Statistical view of fuzzy implication operators

In this section, we consider the expected value of a fuzzy implication, its variance and the distribution of the implication values, assuming that the propositions p and q are independent of each other and the truth values v(p) and v(q) are uniformly distributed across the interval [0,1]. Let a = v(p) and b = v(q). Then the value of the implication I = v(p -> q) is some function of "a" and "b", i.e. I = I(a,b).

Because "a" and "b" are assumed to be uniformly and independently distributed across [0,1], the expected value of the implication is

$$E(I) = \iint_R I(a,b)\, da\,db$$

and its variance is

$$Var(I) = E[(I - E(I))^2] = \iint_R (I(a,b) - E(I))^2\, da\,db$$
$$= E[I^2] - E[I]^2$$

where $R = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}$.

Table 3.1 lists $\overline{E(I)}$ and $\overline{var(I)}$ under these assumptions for the ten operators in definition 2.2. This table provides a benchmark for what to expect for an implication value and the typical spread in values, assuming that the two propositions are completely unrelated.

Table 3.1

operator	expectation E(I)	variance Var(I)
1	1	0
2	1/2 = 0.5	1/4 = 0.25
3	2/3 = 0.667	5/36 = 0.1389
4	3/4 = 0.75	5/48 = 0.1042
4.5	ln 2 = 0.693	$2 - 2 \ln 2 - (\ln 2)^2 = 0.13$
5	5/6 = 0.833	1/18 = 0.0556
5.5	3/4 = 0.75	7/144 = 0.0486
6	2/3 = 0.667	1/18 = 0.0556
7	5/8 = 0.625	3/64 = 0.0469
8	7/12 = 0.583	5/144 = 0.0347

For the illustration, we show the computation process in cases of operators 7 and 8 :

For operator 7, I = max(min(x,y),1-x),

Let $R1 = \{(x,y) : x \le y,\ x < 1/2\}$,

$R2 = \{(x,y) : x \le y,\ x \ge 1/2\}$,

$R3 = \{(x,y) : x \ge y,\ x + y \ge 1\}$,

$R4 = \{(x,y) : x > y,\ x + y \le 1\}$.

Then

$$E(I) = \iint_R I\,dxdy = \iint_{R1}(1 - x)\,dxdy + \iint_{R2} x\,dxdy + \iint_{R3} y\,dxdy + \iint_{R4}(1 - x)\,dxdy$$

$$= 7/24 + 1/12 + 1/8 + 1/8 = 5/8 \quad \text{and}$$

$$Var(I) = \iint_R I^2\,dxdy - E(I)^2 = \iint_{R1}(1 - x)^2\,dxdy + \iint_{R2} x^2\,dxdy + \iint_{R3} y^2\,dxdy + \iint_{R4}(1 - x)^2\,dxdy - E(I)^2$$

$$= 15/64 + 11/192 + 7/96 + 7/96 - 25/64$$
$$= 3/64.$$

For operator 8, $I = \min(\max(1 - x, y), \max(x, 1 - y, \min(y, 1 - x)))$.

Let
$R1 = \{(x,y) : 0 \le x \le 1 - y,\ 1/2 \le y \le 1\}$,
$R2 = \{(x,y) : 0 \le x \le 1/2,\ x \le y \le 1/2\}$,
$R3 = \{(x,Y) : y \le x \le 1 - y,\ 0 \le y \le 1/2\}$,
$R4 = \{(x,y) : 1/2 \le x \le 1,\ 1 - x \le y \le x\}$,
$R5 = \{(x,y) : 1/2 \le x \le 1,\ x \le y \le 1\}$,
$R6 = \{(x,y) : 0 \le x \le 1/2,\ 1 - x \le y \le 1\}$.

Then

$$E(I) = \iint_R I\,dxdy = \iint_{R1} y\,dxdy + \iint_{R2}(1 - y)\,dxdy + \iint_{R3}(1 - x)\,dxdy + \iint_{R4} y\,dxdy + \iint_{R5} x\,dxdy + \iint_{R6}(1 - x)\,dxdy$$

$$= 1/12 + 1/12 + 1/8 + 1/8 + 1/12 + 1/12$$
$$= 7/12 \quad \text{and}$$

$$Var(I) = \iint_R I^2\,dxdy - E(I)^2 = \iint_{R1} y^2\,dxdy + \iint_{R2}(1 - y)^2\,dxdy + \iint_{R3}(1 - x)^2\,dxdy + \iint_{R4} y^2\,dxdy + \iint_{R5} x^2\,dxdy + \iint_{R6}(1 - x)^2\,dxdy$$

$$- E(I)^2 = 11/192 + 11/192 + 7/96 + 7/96 + 11/192 - 49/144$$
$$= 5/144.$$

Let c be a fuzzy implication value, and let F(c) denote the cumulative distribution function, i.e.

$$F(c) = Prob\{I \le c\}.$$

Figure 3.2 is derived from fig. 2.6 and is a contour plot of the same implication values. We find the cumulative

Figure 3.2

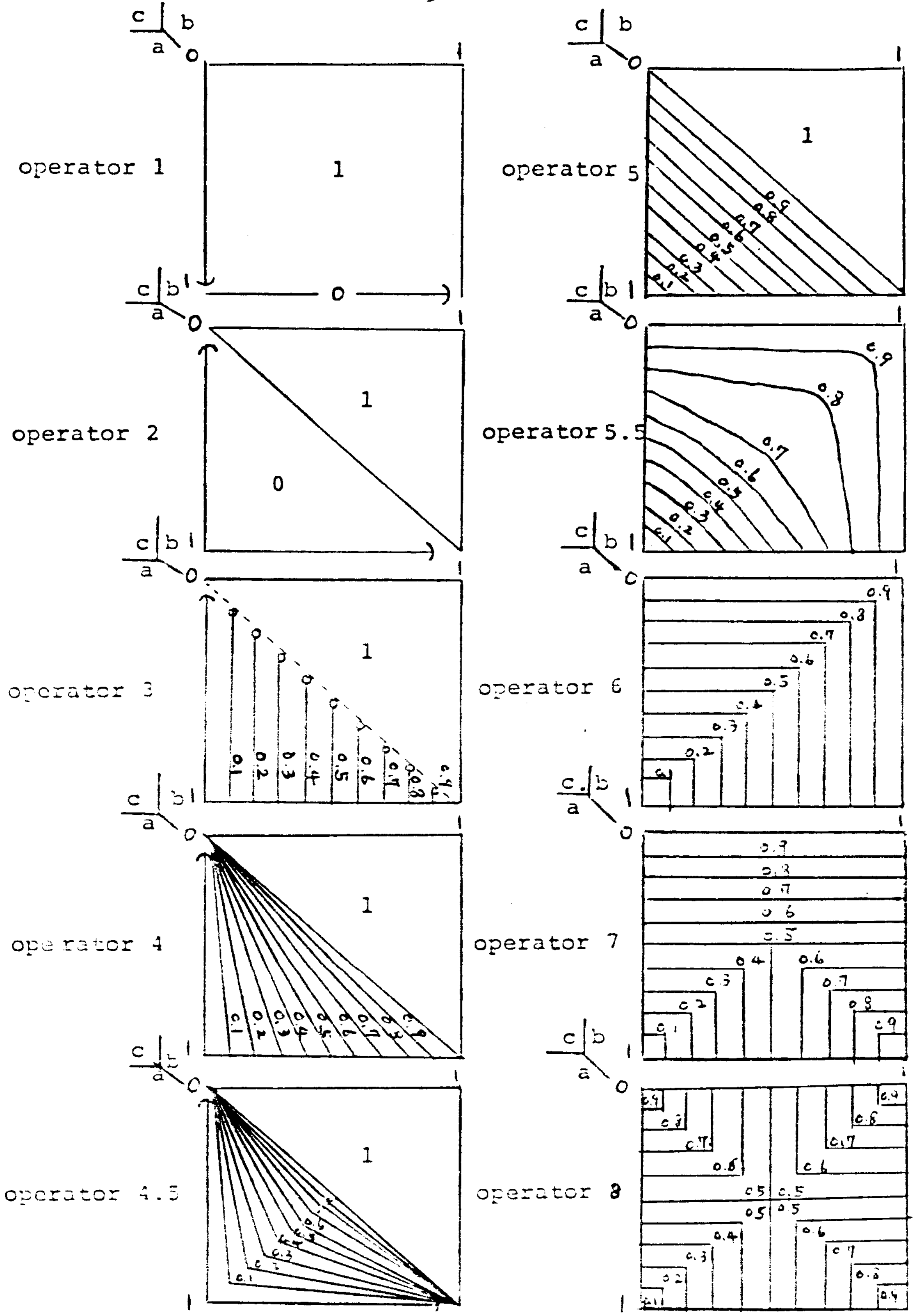
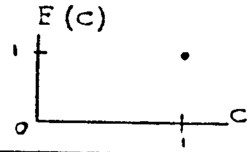
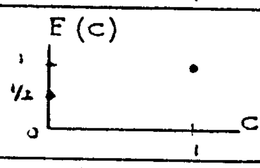
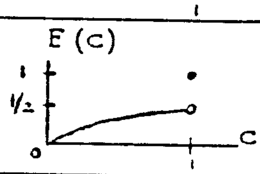
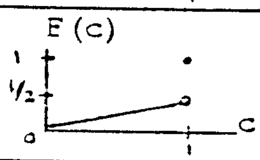
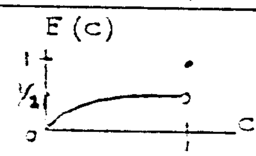
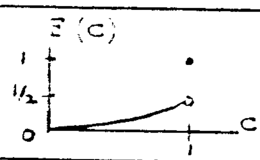
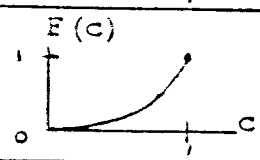
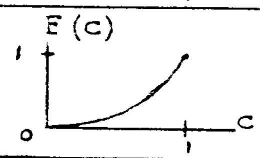
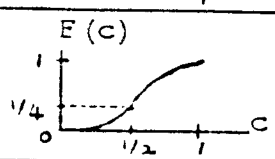
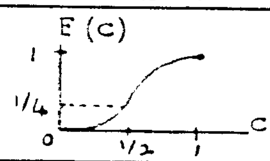

operator 1
operator 2
operator 3
operator 4
operator 4.5
operator 5
operator 5.5
operator 6
operator 7
operator 8

Figure 3.3

operator		cumulative distribution function
1		$F(c) = 1 \quad \text{if } c = 1$
2		$F(c) = \begin{cases} 1/2 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \end{cases}$
3		$F(c) = \begin{cases} -1/2\, c^2 + c & \text{if } c < 1 \\ 1 & \text{if } c = 1 \end{cases}$
4		$F(c) = \begin{cases} c/2 & \text{if } c < 1 \\ 1 & \text{if } c = 1 \end{cases}$
4.5		$F(c) = \begin{cases} c/(c + 1) & \text{if } c < 1 \\ 1 & \text{if } c = 1 \end{cases}$
5		$F(c) = \begin{cases} 1/2\, c^2 & \text{if } c < 1 \\ 1 & \text{if } C = 1 \end{cases}$
5.5		$F(c) = c + (1 - c) \ln (1 - c)$
6		$F(c) = c^2$
7		$F(c) = \begin{cases} c^2 & \text{if } c < 1/2 \\ -(c - 3/2)^2 + 5/4 & \text{if } c \geq 1/2 \end{cases}$
8		$F(c) = \begin{cases} c^2 & \text{if } c < 1/2 \\ -3\,(c - 1)^2 + 1 & \text{if } c \geq 1/2 \end{cases}$

distribution function through the area on the a-b plane with the implication values less than or equal to c and the areas are computed with figure 3.2. Figure 3.3 describes the distribution of c.

We can also consider the probability density function of each fuzzy implication operator by finding the derivative of the distribution function F(c), i.e.,

$$F'(c) = \frac{dF(c)}{dc}.$$

For example, in cases of operators 4.5 and 7, we find F'(c).

For operator 4.5, $F'(c) = \begin{cases} 1/(c+1)^2 & \text{if } c < 1 \\ 1/2 & \text{if } c = 1. \end{cases}$

For operator 7, $F'(c) = \begin{cases} 2c & \text{if } c < 1/2 \\ -2c + 3 & \text{if } c \geq 1/2. \end{cases}$

4. conclusions

In this paper we have investigated some properties of fuzzy implication operators. The difference between the forward implication -> and the backward implication <-, using operator 5, 5.5 and 6, is simply the difference between values of the two propositions in the compound proposition.

The expectation and variance of I is different for the different fuzzy implication operators. If v(p) and v(q) are uniformly distributed and independent of each other, then I is not uniformly distributed for any of the ten kinds of fuzzy implication operators.

5. References.

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