ON ALGEBRAIC STRUCTURES IN THE GENERALIZED FUZZY OPERATIONS

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ABSTRACT

In this paper applied the concepts of triangular norm, which was introduced in [3], we have defined the generalized fuzzy "AND" and "OR" operators. The algebraic structures in the generalized fuzzy operations are discussed.

Keywords: Triangular norm, generalized fuzzy "AND" operator, generalized fuzzy "OR" operator.

I. THE GENERALIZED FUZZY OPERATIONS

Let I be closed unit interval [0,1]. In the following, a, b, c, d etc are arbitrary numbers belonging to I.

Definition 1.1 [3] Let the function of two variables T: $I \times I \rightarrow I$ satisfy the following conditions:

1) $T(a,0)=0$, $T(a,1)=a$
2) monotonicity $T(a,b)\leq T(c,d)$, if $a\leq c$, $b\leq d$
3) symmetricity $T(a,b)=T(b,a)$
4) associativity $T(T(a,b),C)=T(a,T(b,c))$

then T is called the triangular norm or t-norm for short.

Definition 1.2 If $T(a,b)$ is a t-norm then T', defined by $T'(a,b)=1-T(1-a,1-b)$ is called t'-norm for shOrt.

Obviously, T' satisfies following conditions:

5) $T'(a,0)=a, \quad T'(a,1)=1$

6) $T'(a,b)\leqslant T'(c,d)$, if $a\leqslant c, \quad b\leqslant d$

7) $T'(a,b)=T'(b,a)$

8) $T'(T'(a,b),c)=T'(a,T'(b,c))$.

Definition 1.3 We call t-norm T (t'-norm T') the generalized fuzzy "AND" ("OR") operator.

It is easily seen that all fuzzy operators, which was introduced in [2], are respectively T and T' operators.

II. THE STRUCTURE OF $\langle \mathcal{F}, \overset{p}{\cap}, \overset{p}{\cup}, \overset{p}{-}, X, \Phi \rangle$

The set of all fuzzy subsets on universe X is denoted by $\mathcal{F}$. Let $A,B,C,D\in\mathcal{F}$ and $A(X)=a_x$, $B(X)=b_x$, $C(X)=c_x$, $D(X)=d_x$, $x\in X$, $p\geqslant 1$, $q\geqslant 1$, $r>1$, $r\neq 2$. Where p, q, r are real numbers.

Definition 2.1 (1) 1). The p-th power of the fuzzy set A is denoted by $\overset{p}{A}$ and its membership function is defined by $\overset{p}{A}(X)=a_x^p$. The fuzzy sed $\overset{p}{\bar{A}}$ is said to be the complement of order p of A, if $\overset{p}{\bar{A}}(X)=\left(1-a_x^p\right)^{\frac{1}{p}}$, $x\in X$.

2). The intersection of A and B of order p is denoted by $A\overset{p}{\cap}B$ and its membership function is defined by $(A\overset{p}{\cap}B)_{(x)}=$ $=\left(1-\min\left\{1,2-a_x^p-b_x^p\right\}\right)^{\frac{1}{p}}$, $x\in X$.

3). The union of A and B of order p is denoted by $A\overset{p}{\cup}B$ and its membership function is defined by $(A\overset{p}{\cup}B)_{(X)}=$ $\min\left\{1,(a_x^p+b_x^p)^{\frac{1}{p}}\right\}$, $\quad x\in X$.

It quickly follows from above definitions and theorems in

(1) the theorems:

Theorem 2.1 The union of A and B of order p is a t'-norm .

Theorem 2.2 The intersection of A and B of order p is t-norm.

Theorem 2.3 The following propertives hold for $\langle \mathcal{F}, \overset{p}{\cap}, \overset{p}{\cup}, \overset{p}{-}, X, \phi \rangle$:

1). involution law $\overline{\overline{A}^{p}}^{p}=A$

2) commutative laws $A \overset{p}{\cap} B = B \overset{p}{\cap} A$, $A \overset{p}{\cup} B = B \overset{p}{\cup} A$

3) associative laws $A \overset{p}{\cap} (B \overset{p}{\cap} C) = (A \overset{p}{\cap} B) \overset{p}{\cap} C$,

$A \overset{p}{\cup} (B \overset{p}{\cup} C) = (A \overset{p}{\cup} B) \overset{p}{\cup} C$

4).De Morgan's laws $\overline{A \overset{p}{\cap} B}^{p} = \overline{A}^{p} \overset{p}{\cup} \overline{B}^{p}$, $\overline{A \overset{p}{\cup} B}^{p} = \overline{A}^{p} \overset{p}{\cap} \overline{B}^{p}$

5). identity laws $A \overset{p}{\cap} \phi = \phi$, $A \overset{p}{\cap} X = A$, $A \overset{p}{\cup} X = X$, $A \overset{p}{\cup} \phi = A$

6). complementary laws $A \overset{p}{\cap} \overline{A}^{p} = \phi$, $A \overset{p}{\cup} \overline{A}^{p} = X$.

It is easily verified that the idempotent law, absorption law, distributive law , and modular law is no longer true for $\langle \mathcal{F}, \overset{p}{\cap}, \overset{p}{\cup}, \overset{p}{-}, X, \phi \rangle$. Therefore $\langle \mathcal{F}, \overset{p}{\cap}, \overset{p}{\cup}, \overset{p}{-}, X, \phi \rangle$ is a new algebraic structure.

III. THE STRUCTURE OF $\langle \mathcal{F}, \overset{q}{\cap}, \overset{q}{\cup}, \overset{q}{-}, X, \phi \rangle$

Definition 3.1 (1) The intersection of A and B of order q is denoted by $A \overset{q}{\cap} B$ and its membership function is definied by $(A \overset{q}{\cap} B)_{(x)} = \max\left\{0, (a_x^{2q-1} + b_x^{2q-1} - 1)^{\frac{1}{2q-1}}\right\}$, $x \in X$.

(2) The union of A and B of order q is denoted by $A \overset{q}{\cup} B$ and its membership fuction is definied by $(A \overset{q}{\cup} B)_{(x)} =$

$\left(1-\max\left\{0,1-a_x^{2q-1}-b_x^{2q-1}\right\}\right)^{\frac{1}{2q-1}}$, $x\in X$.

(3) The complement $\overset{q}{\bar{A}}$ of A of order q is definied by membership function $\overset{q}{\bar{A}}(x)=(1-a_x^{2q-1})^{\frac{1}{2q-1}}$, $x\in X$.

(4) The q-th power $\overset{q}{A}$ of A is defined by the membership function $\overset{q}{A}(x)=a_x^{2q-1}$, $x\in X$.

Theorem 3.1 (1) When q=1, $A\overset{q}{\cap}B=A\wedge B$, $A\overset{q}{\cup}B=A\,\hat{\vee}\,B$,

$\overset{q}{\bar{A}}=\bar{A}$, $\overset{q}{A}=A$.

(2) When $q=\infty$, $(A\overset{q}{\cap}B)_{(x)}=\begin{cases}0, & \text{when } a_x<1 \text{ and } b_x<1\\ b_x, & \text{when } a_x=1\\ a_x, & \text{when } b_x=1\end{cases}$

$(A\overset{q}{\cup}B)_{(x)}=\begin{cases}0, & \text{when } a_x<1 \text{ and } b_x<1\\ 1, & \text{when } a_x=1 \text{ or } b_x=1\end{cases}$

(3) The intersection of A and B of order q is a t-norm.

(4) The union of A and B of onder q is a t'-norm.

Theorem 3.2 The following propeties hold for $\langle\mathcal{F},\overset{q}{\cap},\overset{q}{\cup},\overset{q}{-},X,\phi\rangle$

1) involution law $\overset{q}{\overline{\overset{q}{\bar{A}}}}=A$

2) commutative laws $A\overset{q}{\cap}B=B\overset{q}{\cap}A$, $A\overset{q}{\cup}B=B\overset{q}{\cup}A$

3) associative laws $A\overset{q}{\cap}(B\overset{q}{\cap}C)=(A\overset{q}{\cap}B)\overset{q}{\cap}C$,

$A\overset{q}{\cup}(B\overset{q}{\cup}C)=(A\overset{q}{\cup}B)\overset{q}{\cup}C$.

4) De Morgan's laws $\overset{q}{\overline{A\overset{q}{\cap}B}}=\overset{q}{\bar{A}}\overset{q}{\cup}\overset{q}{\bar{B}}$, $\overset{q}{\overline{A\overset{q}{\cup}B}}=\overset{q}{\bar{A}}\overset{q}{\cap}\overset{q}{\bar{B}}$

5) identity laws $A\overset{q}{\cap}\phi=\phi$, $A\overset{q}{\cap}X=A$

$A\overset{q}{\cup}\phi=A$ $A\overset{q}{\cup}X=X$.

6) complementary laws $A \overset{q}{\cap} \overset{q}{\bar{A}} = \phi$, $A \overset{q}{\cup} \overset{q}{\bar{A}} = X$.

Similar to II, we can prove that in $\langle \mathcal{F}, \overset{q}{\cup}, \overset{q}{\cap}, \overset{q}{-}, X, \phi \rangle$ idempotent law, absorption law, distributive law and modular law is no longer true, too.

IV. THE STRUCTURE OF $\langle \mathcal{F}, T, T', ^{-}, X, \phi \rangle$

The following theorem can be directly verified.

Theorem 4.1 The following properties hold for $\langle \mathcal{F}, T, T', ^{-}, X, \phi \rangle$:

1) involution law $\bar{\bar{A}} = A$

2) commutative laws $T(A,B)=T(B,A)$, $T'(A,B)=T'(B,A)$

3) associative laws $T(A,T(B,C))=T(T(A,B),C)$

$T'(A,T'(B,C))=T'(T'(A,B),C)$

4) De Morgan's laws $\overline{T'(A,B)}=T(\bar{A},\bar{B})$, $\overline{T(A,B)}=T'(\bar{A},\bar{B})$

5) identity laws $T(A,\phi)=\phi$, $T(A,X)=A$

$T'(A,\phi)=A$, $T'(A,X)=X$.

Definition 4.1 The algebraic structure is called the quasi-De Morgan soft algebra. if it satisfied 1—5 in **theorem 4.1.**

Obviously, $\langle \mathcal{F}, \cdot, \hat{+}, ^{-}, X, \phi \rangle$, $\langle \mathcal{F}, \odot, \oplus, ^{-}, X, \phi \rangle$, $\langle \mathcal{F}, \dot{\varepsilon}, \overset{+}{\varepsilon}, ^{-}, X, \phi \rangle$, $\langle \mathcal{F}, \dot{\gamma}, \overset{+}{\gamma}, ^{-}, X, \phi \rangle$, $\langle \mathcal{F}, \hat{p}, \overset{\vee}{p}, ^{-}, X, \phi \rangle$ and $\langle \mathcal{F}, \hat{\rho}, \check{\rho}, ^{-}, X, \phi \rangle$ [2] are quasi-De Morgan soft algebra.

Theorem 4.2 1) The complementary laws hold for $\langle \mathcal{F}, \odot, \oplus, ^{-}, X, \phi \rangle$ and $\langle \mathcal{F}, \hat{\rho}, \check{\rho}, ^{-}, X, \phi \rangle$.

2) The complementary law is no longer true for $\langle \mathcal{F}, \cdot, \hat{+}, ^{-}, X, \phi \rangle$, $\langle \mathcal{F}, \dot{\varepsilon}, \overset{+}{\varepsilon}, ^{-}, X, \phi \rangle$, $\langle \mathcal{F}, \dot{\gamma}, \overset{+}{\gamma}, ^{-}, X, \phi \rangle$ and $\langle \mathcal{F}, \hat{p}, \overset{\vee}{p}, ^{-}, X, \phi \rangle_{p \neq 1}$

3) In $\langle \mathcal{F}, \cdot, \stackrel{+}{\wedge}, ^{-}, \boxtimes, \phi \rangle$, $\langle \mathcal{F}, \odot, \oplus, ^{-}, \boxtimes, \phi \rangle$,
$\langle \mathcal{F}, \dot{\varepsilon}, \stackrel{+}{\varepsilon}, ^{-}, \boxtimes, \phi \rangle$, $\langle \mathcal{F}, \dot{\gamma}, \stackrel{+}{\gamma}, ^{-}, \boxtimes, \phi \rangle$,
$\langle \mathcal{F}, \stackrel{\perp}{\hat{p}}, \stackrel{\curlyvee}{p}, ^{-}, \boxtimes, \phi \rangle_{p \neq 1}$, $\langle \mathcal{F}, \hat{p}, \check{p}, ^{-}, \boxtimes, \phi \rangle$
idempotent law, absorption law, distributive law and modular law is no longer true.

For comparison of algebraic structures is on the next page.

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Comparison of algebraic structure

	$\langle P, \cap, \cup, ^{-}, \underline{\overline{X}}, \phi \rangle$	$\langle \mathcal{F}, \wedge, \vee, ^{-}, \underline{\overline{X}}, \phi \rangle$	$\langle \mathcal{F}, \overset{p}{\cap}, \overset{p}{\cup}, \overset{p}{-}, \underline{\overline{X}}, \phi \rangle$ $\langle \mathcal{F}, \overset{q}{\cap}, \overset{q}{\cup}, \overset{q}{-}, \underline{\overline{X}}, \phi \rangle$ $\langle \mathcal{F}, \odot, \oplus, ^{-}, \underline{\overline{X}}, \phi \rangle$ $\langle \mathcal{F}, \hat{p}, \overset{\smile}{p}, ^{-}, \underline{\overline{X}}, \phi \rangle$	$\langle \mathcal{F}, \cdot, \hat{+}, ^{-}, \underline{\overline{X}}, \phi \rangle$ $\langle \mathcal{F}, \dot{\varepsilon}, \overset{+}{\varepsilon}, ^{-}, \underline{\overline{X}}, \phi \rangle$ $\langle \mathcal{F}, \dot{\curlyvee}, \overset{+}{\curlyvee}, ^{-}, \underline{\overline{X}}, \phi \rangle$ $\langle \mathcal{F}, \hat{p}, \check{p}, ^{-}, \underline{\overline{X}}, \phi \rangle$
1 commutative law	V	V	V	V
2 associative law	V	V	V	V
3 involution law	V	V	V	V
4 De Morgan's law	V	V	V	V
5 identity law	V	V	V	V
6 idempotent law	V	V	X	X
7 absorption law	V	V	X	X
8 distributive law	V	V	X	X
9 modular law	V	V	X	X
10 complementary law	V	X	V	X

The law hold, denoted by symbol " V ", The law is ne true, denoted by symbol "X".