

THE MAJORITY RULE IN A FUZZY ENVIRONMENT

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Abstract

In this paper, an axiomatic approach to rational decision making in a fuzzy environment is studied. In particular, the majority rule is proposed as a rational way for aggregating fuzzy opinions in a group, when such a group is defined as a fuzzy set.

Keywords: Aggregation, Decisiveness, Fuzzy Majority Rule.

1.- Introduction

Since Arrow (1951) published his Impossibility Theorem, many researchers have tried to get sets of reasonable and logically consistent criteria in order to justify axiomatically democratic decisions. In the context of Fuzzy Set Theory (Zadeh, 1965), Fung and Fu (1975) proposed an axiomatic treatment to the problem of aggregation of fuzzy opinions, but they got a family of aggregation rules too limited. For example, the majority rule was not justified.

Let us describe our problem: we will consider a finite set D of individuals, defined as a fuzzy set

$$\beta : D \longrightarrow [0,1]$$

where $\beta(j)$ represents the degree in which each individual $j \in D$ is "decisor". In our aggregation problem, each group $A \subset D$ defines its preferences through a fuzzy set in a finite set X of actions under study

$$\mu_A : X \longrightarrow [0,1]$$

in such a way that $\mu_A(x)$ represents the degree of acceptance of alternative $x \in X$ by group A .

Let $\mathcal{A}(X)$ be the family of all fuzzy sets in X . Then the opinion of a group A can be represented by a pair

$$(\mu_{A,A}) \in \mathcal{A}(X) \times \mathcal{G}(D) = \mathcal{M}$$

where $\mathcal{G}(D)$ is the family of all groups in D .

We will suppose that $\beta(j) > 0$ for all individuals $j \in D$.

2.- Basic Definitions

Our objective is to get rational aggregations operations which allow us to obtain the opinion of any group A in D from individual opinions.

Definition: An "aggregation" operation is any correspondence which assigns to each pair of opinions, from two disjoint and non-empty groups, the opinion of the union group, verifying associativity and commutativity.

In other words, an aggregation operation is any correspondence

$$\begin{aligned} \odot : \mathcal{M} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (\mu_{A,A}) \odot (\mu_{B,B}) &= (\mu_{A \cup B, A \cup B}) \quad \forall A \cap B = \emptyset \end{aligned}$$

such that

$$\begin{aligned} ((\mu_{A,A}) \odot (\mu_{B,B})) \odot (\mu_{C,C}) &= (\mu_{A,A}) \odot ((\mu_{B,B}) \odot (\mu_{C,C})) \\ (\mu_{A,A}) \odot (\mu_{B,B}) &= (\mu_{B,B}) \odot (\mu_{A,A}) \end{aligned}$$

And some ethical conditions can be imposed to any aggregation operation (see Montero, 1985, for discussion):

(i) Independence of Irrelevant Alternatives: Let $A, B \in \mathcal{G}(D)$ be disjoint and non-empty groups and suppose that

$$\begin{aligned} \mu_A(x) &= \mu'_A(x) & \forall x \in Y \subset X \\ \mu_B(x) &= \mu'_B(x) & \forall x \in Y \subset X \end{aligned}$$

Then

$$\begin{aligned} (\mu_{A,A}) \odot (\mu_{B,B}) &= (\mu_{A \cup B, A \cup B}) \\ (\mu'_{A,A}) \odot (\mu'_{B,B}) &= (\mu'_{A \cup B, A \cup B}) \end{aligned}$$

ought to verify

$$\mu_{A \cup B}(x) = \mu'_{A \cup B}(x) \quad \forall x \in Y \subset X$$

(ii) Non-negative Response: Let $A, B \in \mathcal{G}(D)$ be disjoint and non-empty groups such that

$$\begin{aligned} \mu_A(x) &\geq \mu'_A(x) & \forall x \in X \\ \mu_B(x) &\geq \mu'_B(x) & \forall x \in X \end{aligned}$$

Then

$$\begin{aligned} (\mu_A, A) \odot (\mu_B, B) &= (\mu_{A \cup B}, A \cup B) \\ (\mu'_A, A) \odot (\mu'_B, B) &= (\mu'_{A \cup B}, A \cup B) \end{aligned}$$

ought to verify

$$\mu_{A \cup B}(x) \geq \mu'_{A \cup B}(x) \quad \forall x \in X$$

(iii) Unanimity: Let $A, B \in \mathcal{Q}(D)$ be disjoint and non-empty groups with the same opinion $\mu_A = \mu_B = \mu \in \mathcal{A}(X)$. Then

$$(\mu_A, A) \odot (\mu_B, B) = (\mu, A \cup B)$$

Under these three conditions the following definitions make sense:

Definition: A group G with an opinion μ_G is " δ -decisive" ($0 \leq \delta \leq 1$) regarding an alternative $x \in X$ and a group G' ($G \cap G' = \emptyset$) with an opinion $\mu_{G'}$, such that $\mu_G(x) \neq \mu_{G'}(x)$, when

$$\mu_{G \cup G'}(x) = \delta \cdot \mu_G(x) + (1 - \delta) \cdot \mu_{G'}(x)$$

holds.

Due to conditions we can denote $\delta = \delta_{G, G'}^x(\mu_G(x), \mu_{G'}(x))$ in such a way that

$$\delta_{G, G'}^x(p, q) = 1 - \delta_{G', G}^x(q, p) \quad \forall G \cap G' = \emptyset$$

for any given $p \neq q$.

Definition: The "decisiveness" of an individual $j \in D$ against a group G ($j \notin G$) with opinions $\mu_{\{j\}}$ and μ_G respectively ($\mu_{\{j\}} \neq \mu_G$) is defined by

$$\delta(\{j\}, G) = \inf_{\substack{x \in X \\ \mu_{\{j\}}(x) \neq \mu_G(x)}} \delta_{\{j\}, G}^x(\mu_{\{j\}}(x), \mu_G(x))$$

and his "standardized decisiveness" is defined by

$$\delta(\{j\}, G) / \beta(j)$$

3.- Fuzzy Majority Rule

The main result is the following theorem, which characterizes the fuzzy majority rule:

THEOREM.- Let \odot be any aggregation operation verifying (i), (ii) and (iii). Suppose that such an operation verifies the following condition: each individual always has maximum standardized decisiveness. Then

$$\mu_{G \cup G'}^{\odot}(x) = \left(\sum_{i \in G} \beta(i) \cdot \mu_G^{\odot}(x) + \sum_{i \in G'} \beta(i) \cdot \mu_{G'}^{\odot}(x) \right) / \sum_{i \in G \cup G'} \beta(i)$$

for any $G \cap G' = \emptyset$, in such a way that

$$\mu_D^{\odot}(x) = \sum_{i \in D} \beta(i) \cdot \mu_{\{i\}}^{\odot}(x) / \sum_{j \in D} \beta(j)$$

Proof: Let us suppose a group G and an individual $j \in G$. Since the proposed rule is characterized by

$$\int_{\{j\}, G-\{j\}}^x(p, q) = \beta(j) / \sum_{i \in G} \beta(i) \quad \forall p \neq q, \quad \forall x \in X$$

we will see that for any other rule verifying conditions (i), (ii) and (iii), inside any group G there will be some individual j such that

$$\int_{\{j\}, G-\{j\}}^x(p, q) \leq \beta(j) / \sum_{i \in G} \beta(i)$$

for some $p \neq q$ and some $x \in X$.

In fact, by induction:

a) Let $\text{card}(G-\{j\})=1$, $G-\{j\} = k$, be an unitary group. If we suppose

$$\int_{\{j\}, \{k\}}^x(p, q) > \beta(j) / (\beta(j) + \beta(k))$$

then

$$\begin{aligned} \int_{\{k\}, \{j\}}^x(q, p) &= 1 - \int_{\{j\}, \{k\}}^x(p, q) < \\ &< 1 - \beta(j) / (\beta(j) + \beta(k)) = \\ &= \beta(k) / (\beta(k) + \beta(j)) \end{aligned}$$

and in other case the result follows immediately.

b) Let us assume $\text{card}(G-\{j\})=n$ and $p \neq q$ such that

$$\int_{\{j\}, G-\{j\}}^x (p, q) \leq \beta(j) / \sum_{i \in G} \beta(i)$$

Taking

$$\begin{aligned} \mu_{\{j\}}^{(x)} &= p \\ \mu_{G-\{j\}}^{(x)} &= \mu_{\{k\}}^{(x)} = q \end{aligned}$$

for an individual $k \notin G$, on the one hand we can observe that

$$\mu_{G \cup \{k\}}^{(x)} = \int_{\{j\}, G \cup \{k\} - \{j\}}^x (p, q) \cdot p + \int_{G \cup \{k\} - \{j\}, \{j\}}^x (q, p) \cdot q$$

On the other hand, denoting

$$\begin{aligned} z &= \mu_G^{(x)} = \\ &= \int_{\{j\}, G-\{j\}}^x (p, q) \cdot p + \int_{G-\{j\}, \{j\}}^x (q, p) \cdot q \end{aligned}$$

we observe that

$$\begin{aligned} \mu_{G \cup \{k\}}^{(x)} &= \int_{G, \{k\}}^x (z, q) \cdot z + \int_{\{k\}, G}^x (q, z) \cdot q = \\ &= \int_{G, \{k\}}^x (z, q) \cdot \int_{\{j\}, G-\{j\}}^x (q, p) \cdot p + \\ &\quad + (\int_{G, \{k\}}^x (z, q) \cdot \int_{G-\{j\}, \{j\}}^x (q, p) + \int_{\{k\}, G}^x (q, z)) \cdot q \end{aligned}$$

Therefore,

$$\int_{\{j\}, G \cup \{k\} - \{j\}}^x (p, q) = (1 - \int_{\{k\}, G}^x (q, z)) \cdot \int_{\{j\}, G-\{j\}}^x (p, q)$$

and since

$$\int_{\{j\}, G-\{j\}}^x (p, q) = 0 \quad \Rightarrow \quad \int_{\{j\}, G \cup \{k\} - \{j\}}^x (p, q) = 0$$

we can suppose $z \neq q$. Therefore, if

$$\int_{\{k\}, G}^x (q, z) \leq \beta(k) / \sum_{i \in G \cup \{k\}} \beta(i)$$

the theorem follows immediately. But in the other case,

$$\int_{\{k\},G}^x (q,z) > \beta(k) / \sum_{i \in GU\{k\}} \beta(i) \quad \Rightarrow$$

$$\begin{aligned} \int_{\{j\},GU\{k\}-\{j\}}^x (p,q) &< (1 - \beta(k) / \sum_{i \in GU\{k\}} \beta(i)) \cdot \beta(j) / \sum_{i \in G} \beta(i) \\ &= \beta(j) / \sum_{i \in GU\{k\}} \beta(i) \end{aligned}$$

and the theorem is proved.

Moreover, it is clear that when the group D is homogeneous ($\beta(j)=c$, $c>0$, $\forall j \in D$), the rule verifying all conditions of the previous theorem is defined by

$$\mu_{GU\{G'\}}^x(x) = (\text{card}(G) \cdot \mu_G^x(x) + \text{card}(G') \cdot \mu_{G'}^x(x)) / \text{card}(GU\{G'\})$$

for any $G \cap G' = \emptyset$, in such a way that

$$\mu_D^x(x) = \sum_{i \in D} \mu_{\{i\}}^x(x) / \text{card}(D) \quad \forall x \in X$$

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